Extension of laws in Thompson's group F

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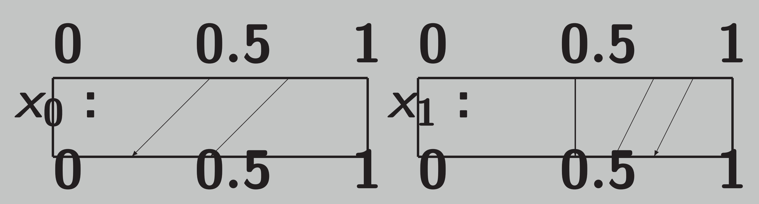
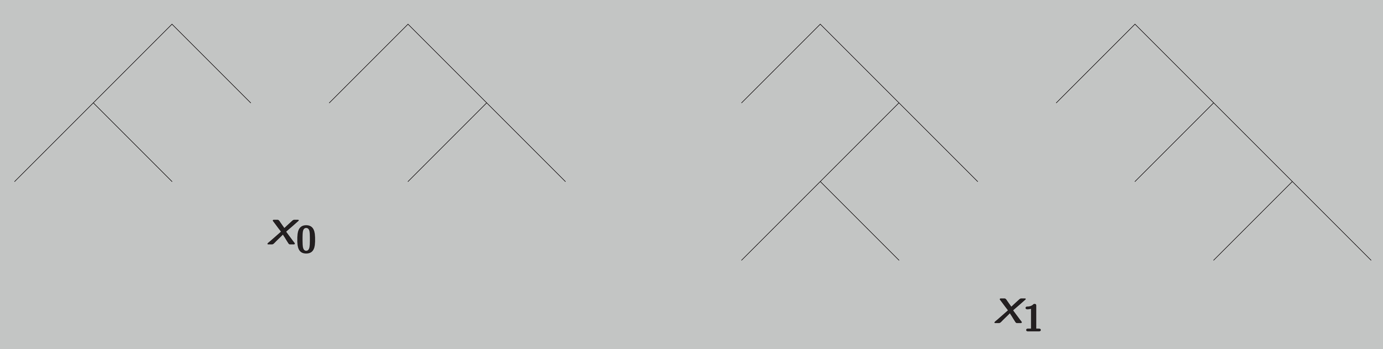
Thompson's group F - equivalent definitions used in researches

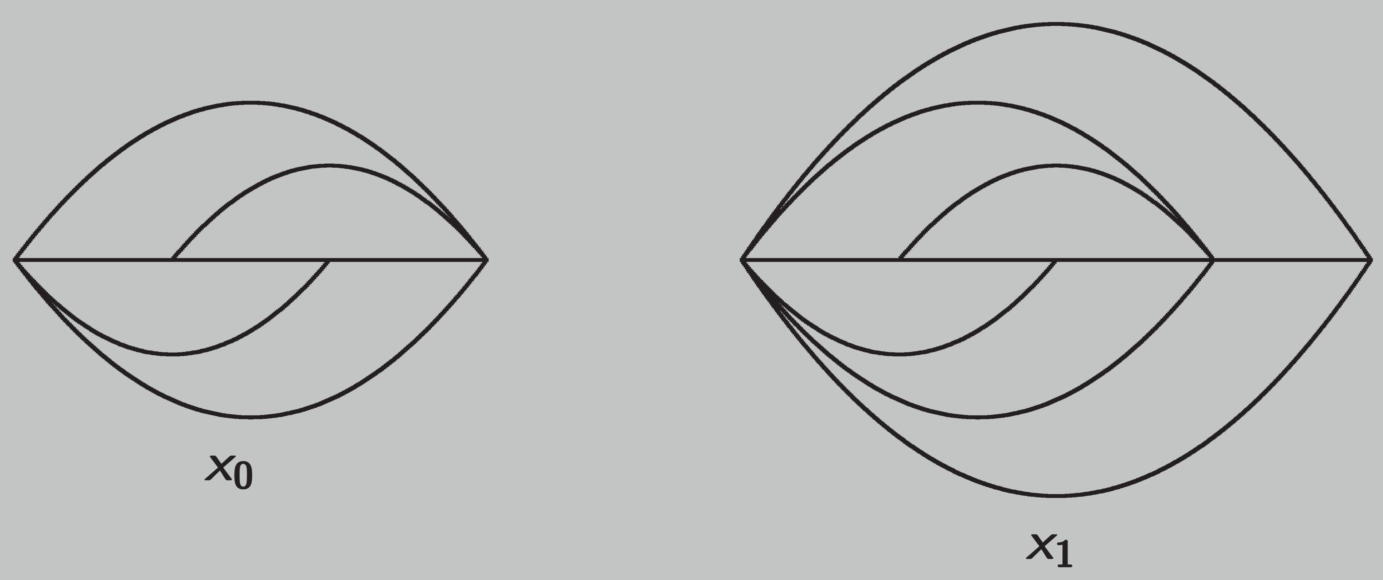
1. Group with presentation $\langle x_0, x_1, x_2, \dots \mid x_n x_k = x_k x_{n+1} \text{ for } k < n \rangle$
2. Group of piecewise linear homeomorphisms from the closed unit interval to itself that are differentiable except at finitely many dyadic rational points and such that derivatives on differentiable points are powers of 2.



3. Group generated by pairs of binary trees corresponding to x_0 and x_1



4. Diagram group of the semigroup $\langle x | x = x^2 \rangle$ where



Laws with constants

- Let G be a group, $H \subseteq G$ and $\mathbb{F}$ be a free group freely generated by $y_1, \cdots, y_z$. Let $w(y_1, \cdots, y_z, g_1, \cdots, g_v)$ be a *reduced* word in the group $G * \mathbb{F}$ with variables $y_1, \cdots, y_z$ and constants $g_1, \cdots, g_v \in G \setminus \{1\}$, where at least one variable and one constant appears. We call $w(y_1, \cdots, y_z, g_1, \cdots, g_v) \equiv 1$ a law with constants (*LC*) satisfied in H if for any $h_1, \cdots, h_z \in H$, $w(h_1, \cdots, h_z, g_1, \cdots, g_v) = 1$.
- $w \equiv u$ is a semigroup law with constants (*SLC*) if both w and u do not contain any variable with negative power.
- Example *SLC* satisfied in S_3:

$$(1\ 2\ 3)x^2(1\ 3\ 2)x^4 = id$$

Bergman's problem for identities

- Let G be a group and S be a semigroup such that $G = gp(S)$. Must each identity satisfied in S be satisfied in G? [1]
- True
  - locally residually finite groups
  - locally graded groups containing no free noncyclic subsemigroups
- False
  - Ivanov's and Storozhev's example [2]

Results: LC in groups

- If a group G satisfies a semigroup law then any *LC* satisfied in G is equivalent to a *SLC*.
- Any group with nontrivial center satisfies *SLC*.
- Cartesian product of groups satisfying *LC* satisfies *LC*.
- Free product of two nontrivial groups where one of them is non-torsion does not satisfy any *LC*.

Results: LC and SLC in F

- (Roland Zarzycki) Consider the standard action of Thompson's group F on $[0,1]$. Suppose we are given two pairwise disjoint open dyadic subintervals $I_i = [p_i, q_i] \subseteq [0,1]$, $i = 1, 2$, and assume that $p_1 < p_2$. Fix any non-trivial $h_1 \in F_{\bar{I}_1}$ and $h_2 \in F_{\bar{I}_2}$ and denote:

$$w^- = [h_1^y, h_2] = y^{-1}h_1^{-1}yh_2^{-1}y^{-1}h_1yh_2$$
$$w^+ = [h_1^{(y^{-1})}, h_2] = yh_1^{-1}y^{-1}h_2^{-1}yh_1y^{-1}h_2$$

  Then the word $w := [w^-, w^+]$ is a law with constants in F.
- (Roland Zarzycki) Consider the standard action of Thompson's group F on $[0,1]$. Suppose we are given four pairwise disjoint closed dyadic subintervals $I_i = [p_i, q_i] \subseteq [0,1]$, $1 \leq i \leq 4$, and assume that $p_1 < p_2 < p_3 < p_4$. Then for any non-trivial $h_1 \in F_{I_1}$, $h_2 \in F_{I_2}$, $h_3 \in F_{I_3}$ and $h_4 \in F_{I_4}$, let

$$w_{14} = [h_1^y, h_4] = y^{-1}h_1^{-1}yh_4^{-1}y^{-1}h_1yh_4$$
$$w_{23} = [h_2^y, h_3] = y^{-1}h_2^{-1}yh_3^{-1}y^{-1}h_2yh_3$$

  Then the word w obtained from $[w_{14}, w_{23}]$ by reduction in $\mathbb{Z} * F$ (we treat the variable y as a generator of $\mathbb{Z}$) is a law with constants in F.

  but...
- Thompson's group F does not satisfy any *SLC*.

Results: Bergman's problem for LC

- Let G be a group and S be a semigroup such that $G = gp(S)$. Must each *LC* satisfied in S be satisfied in G?
- Let $0 < \varepsilon_1 < \varepsilon_2 < \frac{1}{2}$ and let us fix a nontrivial $f \in F_{[\varepsilon_1, \varepsilon_2]}$. The following *LC*

$$f[y_1, y_2] \equiv [y_1, y_2]f \quad (1)$$

  with variables y_1, y_2, and constant f is satisfied in some semigroups generating F but not in F.

References

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