Graphs encoding the generating properties of a finite group

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Ischia Group Theory 2018

March 20th - dedicated to the memory of Michio Suzuki

partially supported by FAPDF - Brazil

The generating graph

The generating graph (M. Liebeck, A. Shalev, 1996)

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- If G is not generated by two elements, then the graph $\Gamma(G)$ is empty. The generating graph encodes significant information only when G is a 2-generator group.
- We introduce and investigate a wider family of graphs which encode the generating property of G when G is an arbitrary finite group.

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The graph $\Gamma^*_{a,b}(G)$

We denote by $\Gamma^*_{a,b}(G)$ the graph obtained from $\Gamma_{a,b}(G)$ by deleting the isolated vertices.

Connectivity

The swap graph

For a d-generator group G, the swap graph $\Sigma_d(G)$ is the graph in which the vertices are the ordered generating d-tuples and in which two vertices $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ are adjacent if and only if they differ only by one entry.

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Roman'kov proved that the free metabelian group of rank 3 does not satisfy this conjecture but no counterexample is known in the class of finite groups.

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Open problem:

to decide whether $\Gamma^*_{a,b}(G)$ is connected when $a + b = d(G)$ and G is unsoluble.

Bounding the diameter of $\Gamma^*_{a,b}(G)$ when G is soluble

Theorem (AC, A. Lucchini)

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$$G = \langle z_1, \ldots, z_a, x_1, \ldots, x_b \rangle = \langle z_1, \ldots, z_a, y_1, \ldots, y_b \rangle.$$

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Corollary

Let G be a finite soluble group and let a and b non-negative integers such that $a + b \geq d(G)$. Then $\operatorname{diam}(\Gamma^*_{a,b}(G)) \leq 4$.

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If G is soluble and $|\operatorname{End}_G(V)| > 2$ for every non-trivial irreducible G-module V which is G-isomorphic to a complemented chief factor of G, then the diameter of the swap graph $\Sigma_d(G)$ is at most $2d - 1$, whenever $d \geq d(G)$.

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If a and b are positive integers, then

$$\lim_{p\to\infty} \mathrm{diam}(\Gamma^*_{a,b}(\mathrm{SL}(2,2^p)^{\tau_{a+b}(\mathrm{SL}(2,2^p))})) = \infty.$$

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We can think that we packaged all the graphs $\Gamma^*_{a,b}(G)$ in a (quite spacious) box but that we did not pay enough attention during this operation and we lost the information to which group G these graphs correspond and the labels a, b.

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Philip Hall observed that the probability of generating a given finite group G by a random t-tuple of elements is given by

$$P_G(t) = \phi_G(t)/|G|^t = \sum_{n \in \mathbb{N}} \frac{a_n(G)}{n^t}, \text{ where } a_n(G) = \sum_{|G:H|=n} \mu_G(H)$$

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- we may determine $P_G(s)$, the uniquely determined Dirichlet polynomial such that for $t \in \mathbb{N}$ the number $P_G(t)$ coincides with the probability of generating G by t random elements. Thus we deduce whether G is soluble or supersoluble, and, for every prime power n, the number of maximal subgroups of G of index n.

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- *Let G be a finite group. We may determine $|\mathrm{Frat}(G)|$ from the knowledge of $\Lambda^*(G)$.*
- *Let G be a finite non-abelian simple group. If H is finite group and $\Lambda^*(H) = \Lambda^*(G)$, then $H \cong G$.*

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Theorem (AC, A. Lucchini)

A finite group G has non-zero spread if and only if $d(G/N) < d(G)$ for every nontrivial normal subgroup N of G, except possibly when $d(G) = 2$.

Thank you!