Algebras in which the group of units is hyperbolic

Victor Bovdi

Department of Mathematical Sciences,
United Arab Emirates University,
United Arab Emirates

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Let (X, ρ) be a metric space with a metric ρ.

For any $a, b, c \in X$, the Gromov product $\langle b, c \rangle_a$ of b and c with respect to $a \in X$ is defined as

$$\langle b, c \rangle_a = \frac{1}{2}(\rho(b, a) + \rho(c, a) - \rho(b, c)).$$

The metric space is called δ-hyperbolic ($\delta \geq 0$) if

$$\langle a, b \rangle_d \geq \min\left\{\langle a, c \rangle_d, \langle b, c \rangle_d\right\} - \delta \qquad (a, b, c, d \in X).$$

Let G be a finitely generated group and let S be a finite, symmetric set of generators for G.

Symmetric set S of generators means that if $g \in S$ then also $g^{-1} \in S$.

The Cayley graph $\mathfrak{C}(G, S)$ of the group G with respect to the set S is the metric graph whose vertices are in one-to-one correspondence with the elements of G.

Their edges (labeled s) of length 1 are joining g to gs
(and gs to g, respectively)
for each $g \in G$ and $s \in S$.

M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

The group G is called a *hyperbolic* group if its Cayley graph

$\mathfrak{C}(G, S)$ is a δ-hyperbolic metric space for some $\delta \geq 0$.

It is well known (see M. Gromov) that this definition does not depend on the choice of the generating set S.

Let KG be the group algebra of a group G over a commutative ring K of characteristic $p \geq 0$.

Let $U(KG)$ be the group of units of the ring KG.

Clearly $G \leq U(KG)$.

The natural question is the following one:

Problem.
When does the group of units $U(KG)$ of the group ring KG of a group G over the commutative ring K with unity is hyperbolic.

For several particular cases this problem was solved in

- $K = \mathbb{Z}$ and G is polycyclic by finite

  S. O. Juriaans, I. B. S. Passi, and D. Prasad. Hyperbolic unit groups. *Proc. Amer. Math. Soc.*, 133(2):415–423 (electronic), 2005.

- G is a finite group, K the ring of integers of a quadratic extension $\mathbb{Q}[\sqrt{d}]$ of the field $\mathbb{Q}$ of rational numbers, where d is a square-free integer $d \neq 1$.

  S. O. Juriaans, I. B. S. Passi, and A. C. Souza Filho. Hyperbolic unit groups and quaternion algebras. *Proc. Indian Acad. Sci. Math. Sci.*, 119(1):9–22, 2009.

- G is a finite group K is a field of a positive characteristic.

  E. Iwaki and S. O. Juriaans. Hypercentral unit groups and the hyperbolicity of a modular group algebra. *Comm. Algebra*, 36(4):1336–1345, 2008.

- Iwaki E, Juriaans S O and Souza Filho A C, Hyperbolicity of semigroup algebras, J. Algebra 319(12) (2008) 5000 - 5015

- Juriaans S O, Polcino Milies C and Souza Filho A C, Alternative algebras with quasihyperbolic unit loops, http://arXiv.org/abs/0810.4544

The idea of proof is the using the following properties of hyperbolic groups

M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

Theorem. If G is a hyperbolic group, then:

(i) $C_\infty \times C_\infty$ does not embed as a subgroup of G;

(ii) if $g \in G$ has infinite order, then $[C_G(g) : \langle g \rangle]$ is finite;

(iii) torsion subgroups of G are finite of bounded order.

(iv) G is virtually free if and only if its boundary has dimension zero;

(v) if G is quasi-isometric to a free group, then G is virtually free. If, moreover, G is torsion-free, then it is free.

I was able to give a complete answer for modular case:

Bovdi, V. Group rings in which the group of units is hyperbolic. (English) J. Group Theory 15, No. 2, 227-235 (2012)

Theorem.
Let KG be the group algebra of a group G over a field K of positive characteristic, such that the torsion part $t(G) \neq \{1\}$. The group of units $U(KG)$ is hyperbolic if and only if when K is a finite field and G is a finite group.

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Theorem.
Let G be a group, such that the torsion part $t(G) \neq \{1\}$. Let K be a commutative ring of $char(K) = 0$ with unity. If the group of units $U(KG)$ of the group ring KG is hyperbolic, then one of the following conditions holds:

(i) $G \in \{C_5, C_8, C_{12}\}$ or G is finite abelian of $exp(G) \in \{2, 3, 4, 6\}$;

(ii) G is a Hamiltonian 2-group;

(iii) $G \in \{H_{3,2}, H_{3,4}, H_{4,2}, H_{4,4}\}$, where $H_{s,n} = \langle a, b \mid a^s = b^n = 1, a^b = a^{-1} \rangle$;

(iv) $G = t(G) \rtimes \langle \xi \rangle$, where $t(G)$ is either a finite Hamiltonian 2-group or a finite abelian group of $exp(t(G)) \in \{2, 3, 4, 6\}$ and $\langle \xi \rangle \cong C_\infty$. Moreover, if $t(G)$ is abelian, then conjugation by ξ either inverts all elements from $t(G)$ or leave them fixed.

Let $K \star G$ be a crossed product of a group G and a commutative ring K.

Let $U(K \star G)$ be the group of units of the ring $K \star G$.

We can ask the following question:

Problem.
When does the group of units $U(K \star G)$ of the crossed product $K \star G$ of a group G and a commutative ring K with unity is hyperbolic.

Thank you for attention!

Grazie per l’attenzione!