Asymptotic linear bounds for the normal covering number of the symmetric groups

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A joint research with C. E. Praeger and P. Spiga

The normal covering number

Let G be a finite non-cyclic group

- A normal covering of G is a set

$$\delta = \{H_i < G : \ i = 1, \ldots, k\}$$

  such that each element in G lies in some conjugate of one of the H_i

- $\gamma(G)$ denotes the minimum size of a normal covering of G and is called the normal covering number of G

- We are interested in $G = S_n$, the symmetric group of degree $n \geq 3$, and in $\gamma(S_n)$ as a function of n

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Types of permutations and normal coverings

- The type of $\psi \in S_n$ is the arithmetic r-partition of n given by the (unordered) list $[x_1, \ldots, x_r]$ of the sizes of the r orbits of ψ in the natural action on $\{1, \ldots, n\}$
- $\delta = \{H_i : \ i = 1, \ldots, k\}$ is a normal covering of S_n

  $$\Updownarrow$$

  for every partition T of n, there exists $i \in \{1, \ldots, k\}$ and $\psi \in H_i$ such that T is the type of ψ

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- The goal: Cover the r-partitions for $1 \leq r \leq n$

Number theory comes into play...

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Known facts

(1) $\gamma(S_n)$ largely depends on the prime factorization

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$$

where p_i are primes with $p_i < p_{i+1}$, $r \geq 1$, $\alpha_i \geq 1$

(2) If $r = 2$ and $n \neq p_1 p_2$ is odd, then

$$\gamma(S_n) = \frac{n}{2}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) + 2 = \frac{\varphi(n)}{2} + 2$$

- Main idea: cover the 2-partitions $[x_1, x_2]$ to get the lower bound; construct a covering to get the upper bound

But the growth is not through the Euler function...

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(3) Deep results on partitions $\Longrightarrow$ there exists $c \in (0, \frac{1}{2}]$ such that

$$cn \leq \gamma(S_n) \leq \frac{2}{3}n$$ linear bounds

If n is even, the constant c can be taken as 0.025 for $n > 792000$ (Magma)

Problem

- The computed values for c are unrealistically small
- Reason: the approximations needed to obtain and then apply the number theoretic results

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Primitive subgroups, n even

Theorem (Guest, Praeger, Spiga - 2016)

Let n be even. If $H < S_n$ is primitive and contains a permutation of type $T = [x_1, x_2, x_3]$, then $H \neq A_n$, and n and T are given by the table below, where q is an odd prime power.

n	T	Comments
10	$[2,4,4]$, $[1,3,6]$	
22	$[4,6,12]$, $[1,7,14]$, $[2,10,10]$	
26	$[2,12,12]$	
28	$[4,12,12]$	
36	$[12,12,12]$	
$\frac{q^d-1}{q-1}$	$\left[\frac{q^d-1}{3(q-1)}, \frac{q^d-1}{3(q-1)}, \frac{q^d-1}{3(q-1)}\right]$	$d \geq 2$ even
$\frac{q^d-1}{q-1}$	$\left[\frac{q^{d_1}-1}{q-1}, \frac{q^{d_2}-1}{q-1}, \frac{(q^{d_1}-1)(q^{d_2}-1)}{q-1}\right]$	$d_1, d_2 \geq 1$, $d = d_1 + d_2 \geq 2$ even, $\gcd(d_1, d_2) = 1$

- $T = [x_1, x_2, x_3]$ is coprime if $gcd(x_1, x_2, x_3) = 1$

Proposition (BPS - 2017)

The number of coprime 3-partitions covered by proper primitive subgroups of S_n is at most $\frac{(\log_3(n))^2 + 3\log_3(n)}{8}$

A previous result (BPS - 2013)

The number of coprime 3-partitions of n covered by imprimitive subgroups of S_n is at most $2n^{3/2}$

A known fact (M. E. Bachraoui - 2008)

The number of coprime 3-partitions of $n \geq 4$ is

$$\frac{1}{12}n^2 \prod_{p|n}\left(1 - \frac{1}{p^2}\right) > \frac{1}{12}n^2\zeta(2)^{-1} = \frac{n^2}{2\pi^2}$$

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Intransitive subgroups, n even

- The new ideas

1. The main role in a normal covering of S_n is played by the maximal intransitive subgroups $S_k \times S_{n-k}$, with $1 \leq k < \frac{n}{2}$

2. Proposition (BPS - 2017)

The number of 3-partitions covered by ℓ maximal intransitive subgroups is at most $\frac{\ell}{2}(n - \ell + 1)$

$\Downarrow$

3. To cover at least all the coprime 3-partitions of n we need ℓ maximal intransitive subgroups of S_n such that

$$\frac{\ell}{2}(n - \ell + 1) \geq \frac{n^2}{2\pi^2} - \frac{(\log_3(n))^2 + 3\log_3(n)}{8} - 2n^{3/2}$$

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Main result

Since $\gamma(S_n) \geq \ell$, by the previous inequality solved in $\frac{\ell}{n}$, we get

Theorem (BPS - 2017)

Let n be even. Then

$$\gamma(S_n) \geq n\frac{1-\sqrt{1-4/\pi^2}}{2} + o(n)$$

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- $\frac{1-\sqrt{1-4/\pi^2}}{2} \approx 0.1144$ is greatly larger than 0.025...

- The function $o(n)$ is known $\Longrightarrow \forall \varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ explicitly computable such that

$$\gamma(S_n) \geq n \left[\frac{1 - \sqrt{1 - 4/\pi^2}}{2} - \varepsilon \right]$$

for all $n \geq n_\varepsilon$ even

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Thank you for your attention!

n odd

- Good news
  - We can deal with 4-partitions. They avoid the alternating group
  - The primitive subgroups of S_n covering the 4-partition are known (Guest, Praeger, Spiga - 2016)
- Bad news
  - The full control on the number of 4-partitions inside ℓ intransitive maximal subgroups of S_n is hard
  - There exists no exact formula for the number of the comprime 4-partitions (but we have bounds)
  - The work is in progress...

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