Involutions in the multiple holomorph of a group

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Holomorphs

The Holomorph

Let G be a group, $S(G)$ be the group of permutations on the set G, and

$$\begin{aligned} \rho : G &\to S(G) \\ g &\mapsto (x \mapsto xg) \end{aligned}$$

be the right regular representation. Then

$$N_{S(G)}(\rho(G)) = \mathrm{Aut}(G)\rho(G) = \mathrm{Hol}(G)$$

is the holomorph of G.

More generally, if $N \leq S(G)$ is a regular subgroup, then

$$N_{S(G)}(N)$$

is isomorphic to the holomorph of N.

Same Holomorph

So if $N \leq S(G)$ is regular, we may say that G and N have the same holomorph if

$$N_{S(G)}(N) = N_{S(G)}(\rho(G)) = \text{Aut}(G)\rho(G) = \text{Hol}(G).$$

If G and N have the same holomorph, and are isomorphic, then $\rho(G)$ and N are conjugate under an element of

$$N_{S(G)}(\text{Hol}(G)) = N_{S(G)}(N_{S(G)}(\rho(G))),$$

the multiple holomorph of G.

$T(G)$

The group

$$T(G) = N_{S(G)}(\mathrm{Hol}(G)) / \mathrm{Hol}(G)$$

acts regularly on the set

$$\begin{aligned}\mathcal{H}(G) = \{N \le S(G) : {} & N \text{ is regular,} \\ & N_{S(G)}(N) = \mathrm{Hol}(G), \text{ and} \\ & N \cong G\}.\end{aligned}$$

It appears handier to compute first the set

$$\mathcal{J}(G) = \{ N \le S(G) : N \text{ is regular, and } N \trianglelefteq \mathrm{Hol}(G) \} \supseteq \mathcal{H}(G),$$

then check which elements of $\mathcal{J}(G)$ are in $\mathcal{H}(G)$, and finally compute $T(G)$.

Describing the regular normal subgroups of the holomorph

Regular normal subgroups of the holomorph

A regular normal subgroup $N \trianglelefteq \mathrm{Hol}(G) = \mathrm{Aut}(G)\rho(G)$ can be characterized in two ways. First by the map

$$\gamma : G \to \mathrm{Aut}(G), \qquad N \ni \nu(g) = \gamma(g)\rho(g),$$

where $1^{\nu(g)} = g$. Such γ are characterized by

$$\gamma(gh) = \gamma(h)\gamma(g), \qquad \text{and} \qquad \gamma(g^{\beta}) = \gamma(g)^{\beta} \quad \text{for } \beta \in \mathrm{Aut}(G).$$

N can be also characterized by the group operation

$$g \circ h = g^{\gamma(h)}h, \qquad \text{for which } \nu : (G, \circ) \to N \text{ is an isomorphism,}$$

which is characterized by

$$(gh) \circ k = (g \circ k)k^{-1}(h \circ k) \qquad \text{and} \qquad \mathrm{Aut}(G) \leq \mathrm{Aut}(G, \circ).$$

See under skew right braces.

Abelian Groups

In

A. C. and F. Dalla Volta
The multiple holomorph of a finitely generated abelian group
J. Algebra **481** (2017), 327-347

we have redone the work of

W. H. Mills
Multiple holomorphs of finitely generated abelian groups
Trans. Amer. Math. Soc. **71** (1951), 379-392

Abelian groups and commutative rings

When $(G, +)$ is abelian (additively written), one can rephrase the operation "$\circ$" in terms of an operation "$\cdot$" that makes $(G, +, \cdot)$ into a commutative ring. The condition

$$\mathrm{Aut}(G) \leq \mathrm{Aut}(G, \circ)$$

translates into the study of the (very restricted) commutative rings $(G, +, \cdot)$ such that

> *every automorphism of the group $(G, +)$ is also an automorphism of the ring $(G, +, \cdot)$.*

Here the $T(G)$ are elementary abelian 2-groups, and small: $|T(G)| \leq 4$.

(Many) involutions

There will be (many) involutions

Let G be a non-abelian group, and consider the left regular representation λ. Then

$$N_{S(G)}(\lambda(G)) = \mathrm{Hol}(G) = N_{S(G)}(\rho(G)).$$

Here the group operation "$\circ$" associated to the regular subgroup $\lambda(G) \trianglelefteq \mathrm{Hol}(G)$ is

$$x \circ y = yx,$$

that is, $(G, \circ)$ is the opposite group of G.

$\mathrm{inv} : x \mapsto x^{-1}$ yields an involution in $N_{S(G)}(\mathrm{Hol}(G))$, and thus in $T(G) = N_{S(G)}(\mathrm{Hol}(G))/\mathrm{Hol}(G)$, as

$$\rho(G)^{\mathrm{inv}} = \lambda(G) \leq \mathrm{Hol}(G).$$

Any regular subgroup $N \leq S(G)$ will yield an involution in $T(G)$, although all these involutions need not be distinct.

Perfect groups

A. C. and F. Dalla Volta

Groups with the same holomorph as a finite perfect group

arXiv:1612.03573

If G is a finite, perfect, centreless group, then we obtain all regular $N \trianglelefteq \mathrm{Hol}(G)$ (via the "$\circ$" operation) as follows. Let the Krull-Remak decomposition of G as an $\mathrm{Aut}(G)$-group be

$$G = L_1 \times \cdots \times L_n.$$

Select *any* $m \leq n$ of the L_i, say,

$$H = L_1 \times \cdots \times L_m, \qquad K = L_{m+1} \times \cdots \times L_n.$$

We obtain $(G, \circ)$ by replacing H with its opposite: generalizes the relation between right and left regular representations.

Here, too, $T(G)$ is elementary abelian, of size 2^n.

A quasisimple question

The case of a finite perfect group G with $Z(G) \neq 1$ leads to the following question.

Is there a (family of) quasisimple group(s) Q such that

- $Z(Q)$ is not elementary abelian, and
- $\mathrm{Aut}(Q)$ acts trivially on $Z(Q)$?

We have perfect examples Q, which are good enough to exhibit the pathologies of this case.

Finite p-group of class two: more than involutions

Not only involutions

A. C.
The Multiple Holomorphs of Finite p-Groups of Class Two
arXiv:1801.10410

For $p > 2$ a prime, consider the group

$$\mathcal{G}(p) = \langle\, x, y : x^{p^2}, y^{p^2}, [x, y] = x^p \,\rangle$$

of order p^4 and nilpotence class 2.

Then

$$|\, T(\mathcal{G}(p))| = p(p-1),$$

and for $p > 3$ this group is not generated by involutions.

From 2 to $p-1$

In finite p-groups G of nilpotence class two there is an element of order $p-1$ in $T(G)$.

Such an element is given by the d-th power map $g^{\vartheta_d} = g^d$, for d of multiplicative order $p-1$ modulo $\exp(G)$. In fact if $\iota(x)$ denotes the conjugation by x, we have

$$\rho(g)^{\vartheta_d} = \iota(g^{(1-d)/2})\rho(g^d) \in \mathrm{Aut}(G)\rho(G) = \mathrm{Hol}(G),$$

so that $\vartheta_d \in N_{S(G)}(\mathrm{Hol}(G))$ induces an element of order $p-1$ in

$$T(G) = N_{S(G)}(\mathrm{Hol}(G))/\,\mathrm{Hol}(G).$$

More examples (using $\gamma(g^\beta) = \gamma(g)^\beta$)

- $|T(G)| = p-1$
- $|T(G)|$ contains a subgroup of order $(p-1)p^{\binom{n}{2}\binom{n+1}{2}}$, where n is the minimal number of generators of G.

That’s All, Thanks!