Report on **WORK IN PROGRESS** about

GROUPS with RESTRICTIONS on SUBGROUPS 'which are not' COMMENSURABLE with a NORMAL SUBGROUP

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A group G is said to have the **weak minimal condition** on $\mathcal{X}$-subgroups

Min-∞-$\mathcal{X}$

if there are NO infinite descending sequences $H_0 > H_1 > \ldots H_i > \ldots$ of $\mathcal{X}$-subgroups of G for which all indices $|H_i : H_{i+1}|$ are infinite.

Fact: A soluble (by-finite) Minimax [a] group has Min-∞.

[a] i.e. it has a finite series whose factors satisfy Max or Min

Many results on generalized soluble groups Min-∞-$\mathcal{X}$ have appeared.

(Zaitsev/Baer 1968) Let G be a locally soluble group:
If G has the weak minimal condition Min-∞ (for all subgroups) ,
then G is a soluble Minimax group.

- L. Kurdachenco (since 1979) has studied Min-∞-normal.
- M. Dixon, M. Ferrara, M. Trombetti (IGT 2018, Poster Session) consider Min-∞-....

Phillips and Wilson, J. Algebra 51 (1978)

A locally graded[a] group with the minimal condition on non-normal subgroup has either the minimal condition or all subgroups are normal.

[a]each finitely generated non-trivial subgroup has a subgroup with finite index > 1

Consider the following **dicotomy** (D) for a "generalized" soluble group G:

(D) G has Min-∞-not$\mathcal{X}$ iff either G is Minimax or all subgroups have $\mathcal{X}$.

L. A. Kurdachenko and V. E. Goretskii (1987)

(D) HOLDS for locally (soluble-by-finite) groups and $\mathcal{X}$=normality.

L. A. Kurdachenko - H. Smith (1997)

(D) HOLDS for generalized radical [a] groups and $\mathcal{X}$=subnormality [b]

[a]a group is called **generalized radical** if it has an ascending series of normal subgroups the factors of which are locally (nilpotent or finite)

[b]A subgroup H is **subnormal** if there is a finite series $H = H_0 \lhd H_1 \lhd \dots H_n = G$.

the POSET of commensurability classes

The relation of *almost containment* beetween subgroups H and K defined by $|H : (H \cap K)| < \infty$ is a preorder (i.e. transitive and simmetric) and generates a relation of **equivalence called commensurability**.

H and K are commensurable if $H \cap K$ has finite index in both H and K.

FACT: (when $\mathcal{X}$ is closed under intersections of 2 subgroups)
the weak minimal condition on $\mathcal{X}$ -subgroups

Min-∞-$\mathcal{X}$

is just the minimal condition

for the poset of the of commensurabilty classes of $\mathcal{X}$-subgroups.

Specht-Heineken (Math. Nachr. 134 (1987)

studied locally nilpotent and locally finite groups
with only finitely many commensurability classes of subgroups.

H. Heineken and L. Kurdachenko (Algebra Colloq. 4, 1997)

studied locally nilpotent groups and metabelian groups
with only *finitely many commensurability classes of normal subgroups*.

A subgroup H is called a **cn-subgroup** if it is commensurable with a normal subgroup. i.e.
(**cn**) there is $N \lhd G$ such that $|HN : (H \cap N)| < \infty$.

This concept generalizes both the classical notions

(**nn**) H is called nearly-normal if $|H^G : H| < \infty$
(**cf**) H is called normal-by-finite (core-finite) subgroup if $|H : H_G| < \infty$.

Note that if H is **cn**, then H is 2-sbyf,
i.e. it has a subgroup with finite index which is subnormal with defect ≤ 2 subgroup in G, namely $(H \cap N)_N$.

B.H. Neuman, 1955

NN) if all subgroups of the group G are **nn** (nearly normal),
then G is **finite-by-abelian**

Further, if G has the property PILF (Periodic Images are Locally Finite), then

J.T.Buckley, J.C.Lennox, B.H.Neumann, H.Smith, J.Wiegold, 1995

CF) if all subgroups of the PILF group G are **cf** (normal-by-finite),
then G is **abelian-by-finite**

C.Casolo, U.D., S.Rinauro, J. Algebra 496 (2018)

CF) if all subgroups of the PILF group G are **cn**,
then G is **finite-by-abelian-by-finite**.
Moreover, for each prime there is a soluble p-group in which all subgroup are cn, but the group is neither finite-by-abelian nor abelian-by-finite.

Problem: find sufficient conditions for a **cn**-subgroup $H \sim N \triangleleft G$ to be cf. Since $H \cap N \leq_f H$ we may assume $H = H \cap N \leq N$, that is H is nn.

Let H a **cn**-subgroup of a group G, then H is **cf**, provided one of the followiing holds.
0) G is abelian-by-finite
1) H has finitely many G-conjugates .
2) H is finitely generated,
3) H has the property that its Bounded Images are Finite

A group has finite rank r if all finitely generated subgroups can be generated by r elements (and r is the minimum such a natural number)

Problem (see F. de Giovanni, J. Math. Sci. (N.Y.) 199 (2014))

Let G be a generalized radical group with infinite rank.
(**SIR**) If every Subgroup H of G with Infinite Rank has the property $\mathcal{X}$, shall all subgroups have $\mathcal{X}$?

(**SIR**) holds for very many properties $\mathcal{X}$, in particular for $\mathcal{X} =$

M. De Falco, F. de Giovanni, C. Musella, Publ. Mat. 58 (2014)

an) almost normal (:= with a finite number of conjugates)
nn) nearly normal (core-finite)
cf) normal-by-finite,

UD + SR

cn) commensurable with a normal subgroup.

FMCC-$\mathcal{X}$: **only Finitely Many Conjugacy Classes of $\mathcal{X}$-subgroups**

- Tarski groups have FMCC
- locally graded FMCC-groups must be finite.

Franciosi, de Giovanni, Korea 1995

Let G be a locally graded group.
nn) G has finitely many conjugacy classes of nn-subgroups iff either G' is finite or G is cyclic-by-finite.

Let G be group whose Periodic Images are Locally Finite.

De Falco, Musella, de Giovanni, PAMS 125,1997

cf) G has finitely many conjugacy class of subgroups which are not cf iff all subgroups are cf.

UD+SR

cn) G has finitely many conjugacy class of subgroups which are not cn iff all subgroups are cn.

Here one considers the *frame*, i.e. the set of coniugacy classes, which is a poset, provided $H > H^x$ do NOT happen for $x \in G$ and $H \leq G$

A locally graded group with Min-(non-normal) either has the minimal condition Min or all subgroups are normal. (Phillips and Wilson)

Franciosi, de Giovanni, Korea 1995

Let G be a group with Min-non- **nn** (nearly-normal)
nn') if G is locally finite, then either G' is finite or G has Min .
nn") if G is non-periodic group, then either G' is finite or G has a finite normal subgroup E such that $\bar{G} = G/E = \bar{J} \times \bar{D}$ where $\bar{J}$ is the direct product of finitely many Prufer groups and $\bar{D}$ is infinite dihedral.

Let G be group whose Periodic Images are Locally Finite.

Franciosi, de Giovanni, Proc. Amer. Math. Soc. 125 (1997)

cf) If G has Min-non- **cf** (core-finite),
then either G has Min or all its subgroups are **cf**.

UD+SR

cn) If G has Min-non- **cn** (commensurable with a normal subgroup),
then either G has Min or all its subgroups are **cn**.

Theorem

Let G be a generalized radical group, then:

1) every subgroup of infinite rank of G is **cn** *iff either G has finite rank or all subgroups are* **cn** *.*

2) G has finitely many conjugacy class of subgroups which are not **cn** *iff all subgroups are* **cn** *.*

*3) G has the minimal condition on non-***cn***-subgroups iff either G has the minimal condition on all subgroup or all subgroups are* **cn** *.*

THE END?

This is not the end... :-) ... the work is still in progress.
Now we are working on Min-∞-(non-cn),
which is also the minimal condition on commensurability classes with no normal members
Anyway ...

THANK YOU

for your attention