Symmetric groups and fixed points on modules

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Dedicated to the memory of

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1. Introduction

References:

[1] GG - J. Lynd, Invent. Math. 2016

[2] GG, Contemporary Math. 688, 2017

Consider the prime 2

Easy to work with: Two elements of order 2 generate abelian or dihedral group

Hard to work with: Examples later

(GpS) Assume G finite group, p prime,

S **non - identity** Sylow p-subgroup of G

LOCAL ANALYSIS

Info. about $N_G(T)$ for some subgrps T,
$1 < T \leq S$, gives global information about G

Especially useful if Info about $N_G(T)$ for **one** T,

$1 < T \leq S$, gives global information about G

EXAMPLE (Burnside) If S is contained in center of $N_G(S)$, then G has normal p-complement.

(Here, p is any prime; in most theorems for **one** T, p is odd and there is counterexample for p = 2.)

2. Fixed points on modules

Assume G is a group of automorphisms of finite abelian p-group D

Let ~~C (D)~~ $C_D(G)$ = fixed point subgrp of ~~G~~ D under ~~D~~ G
= $\{x \in D \mid x^g = x$ for all g in G$\}$

Given elementary abelian p-subgroup A of G,

A is an **offender** if $|A| \geq |D/C_D(A)|$, e.g. A = 1;

non-trivial offender if also A > 1;

minimal offender if A is minimal non-trivial offender under inclusion

Theorem 1 (GG, 1971) Assume G contains a non-trivial offender . If p is odd then

(A) There exists a normal subgroup T of S generated by non-trivial offenders such that

$C_D(G) = C_D(N_G(T))$.

Counterexample for p = 2:

D = Klein 4-group, G = Aut D = S_3

3. The Martino - Priddy Conjecture

(Connection between top. space associated with G and properties of S in G)

Proof and extensions by J. Martino, S. Priddy, B. Oliver, A. Chermak; assume classification of finite simple groups (CFSG)

(GG - Lynd, 2016) Don't need CFSG in proof; used Theorem 1 for p odd, and proved and used Theorems 2,3 below for p = 2

Theorem 2 Suppose G contains a non - trivial offender and (**A**) is false. Then

(**B**) $p = 2$ and every minimal offender has order 2 (and other conditions)

Theorem 3 In Theorem 2, if $O_2(G) = 1$ and G is generated by the minimal offenders in (**B**), then G is a direct product of symmetric groups of odd degree $G = S_{m_1} \times S_{m_2} \times \ldots \times S_{m_k}$, for odd $m_1, m_2, \ldots, m_k \geq 3$.

Note: Theorems 2, 3 true for counterexample $G = S_3$.