Integral Forms in Vertex Operator Algebras

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1 Introduction

Definition 1.1. An integral form in an algebra in characteristic 0 is the $\mathbb{Z}$-span of a basis which is closed under the product.

Example 1.2. (1) $Mat_{n\times n}(\mathbb{Z}) \leq Mat_{n\times n}(\mathbb{C})$; (2) In a simple finite dimensional complex Lie algebra, the $\mathbb{Z}$-span of a Chevalley basis is an integral form.

2 VOAs

Definition of VOA is too long to give here. Standard reference for VOA theory: Vertex Operator Algebras and the Monster, by Frenkel, Lepowsky, Meurman; see definition p.244.

Short version: $V = \bigoplus_{n\geq 0} V_n$, a graded vector space in characteristic 0 with each $dim(V_i)$ finite; vacuum element $\mathbf{1} \in V_0$, Virasoro element $\omega \in V_2$; a linear monomorphism $Y : V \to End(V)[[z^{-1}, z]]$, written $Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$.

For each $n \in \mathbb{Z}$, there is a product $a, b \mapsto a_n b$ (meaning the endomorphism a_n applied to b), giving a ring (V, n^{th}).

A vertex algebra (VA) is a generalization of VOA to graded modules over a commutative rings of scalars. It has a vacuum element but not necessarily a Virasoro element. Our examples will be over finite fields or the integers.

Example 2.1. For an even integral lattice L, there is a lattice type VOA V_L which, as a graded vector space, has the form $\mathbb{S}(\hat{H}) \otimes \mathbb{C}[L]$. Here, $\mathbb{S}$ means symmetric algebra of a vector space, $H := \mathbb{C} \otimes_{\mathbb{Z}} L$ and $\hat{H}$ means $H_1 \oplus H_2 \oplus \dots$ where H_k is a copy of H declared to have degree k. Finally $\mathbb{C}[L]$ is the group algebra of the abelian group L, with basis e^α, for $\alpha \in L$.

Example 2.2. The Moonshine VOA $V^\natural$ is a twisted version of V_Λ, where Λ is the Leech lattice (rank 24, determinant 1, minimum norm 4). It has $Aut(V^\natural) \cong \mathbb{M}$, the Monster.

Definition 2.3. An *integral form* R in a vertex operator algebra $V = \bigoplus_{n \in \mathbb{Z}} V_n$ with a nondegenerate symmetric bilinear form is the $\mathbb{Z}$-span of a basis which is closed under all the VOA products and for all n, $R \cap V_n$ is an integral form of the vector space V_n; also R must contain the vacuum element and a positive integer multiple of the Virasoro element.

So, an integral form in a VOA is a vertex algebra over the ring of integers.

Chongying Dong and I studied the following question. Given a finite group G in $Aut(V)$, is there an integral form in V which is stable under G? We have some general sufficient conditions.

Our main applications: (1) for lattice type VOAs, (L of rank r) there is a G-invariant integral form where G has the form $2^r.O(L)$. (One description: it is generated by the $\mathbb{Z}$-span of the components of $Y(e^\alpha, z)\mathbf{1}$ for $\alpha \in L$.)

(2) For the Moonshine VOA $V^\natural$, we proved that there is a Monster-invariant integral form. (This is created by a kind of averaging, and is not described explicitly.)

Remark 2.4. We learned after our proof was written, that in the 80s, Borcherds had asserted the existence of an integral form for lattice type VOAs.

He observed that there is a Monster-invariant $\mathbb{Z}[\frac{1}{2}]$-form in $V^\natural$ but claimed nothing about a form over $\mathbb{Z}$.

3 Lattice-integrality

If J is an integral form in a VOA, it inherits a symmetric bilinear form from the VOA. We say J is *lattice integral* if $\langle x, y \rangle \in \mathbb{Z}$ for all $x, y \in J$. It is unclear when the restriction of the form to J is integral-valued (or can be multiplied by a scalar to become integral valued).

Dong and I gave two sufficient conditions to prove lattice integrality. (1) We showed that it is integral valued whenever the integral form J is generated by quasi-primary vectors (in VOA theory, this means vectors annihilated by a certain operator $L(1)$). This criterion applies to the Monster-invariant form we built earlier. (2) We gave an averaging-type argument.

4 Classical lattice type VOAs

Take the case where L is a root lattice of type ADE, and let $V := V_L$. Then $(V_1, 0^{th})$ is a copy of the Lie algebra associated to L. *If J is our integral form, $J \cap V_1$ is the $\mathbb{Z}$-lattice spanned by a Chevalley basis! In fact, this J is spanned by a set of elements which generalizes "Chevalley basis".* It turns out that a Chevalley group can be defined on V_L with the standard generators $x_r(\pm 1)$ fixing the integral form.

We can also take any commutative associative ring R, get a VA $R \otimes J$ over R, *the classical VA of type L over R.* We also get an action of the Chevalley group of type L over R on $R \otimes J$ as VA automorphisms.

When R is a field we get all Chevalley groups of types ADE (with graph outer automorphisms) as full automorphism groups of these VAs over R. We also defined VA over R for types BCGF. This gives the Steinberg variations (twisted Chevalley groups) acting on VAs and being essentially the full automorphism groups.

We would like to find a series of VAs whose automorphism groups are essentially the Ree and Suzuki groups but have not (yet) done so.

Our construction gives an infinite dimensional graded modules for each Chevalley group and Steinberg variation over its field of definition. These modules may be a good opportunity for study of representation theory (Ext, indecomposables, etc.).

5 The degree 2 component of a VOA

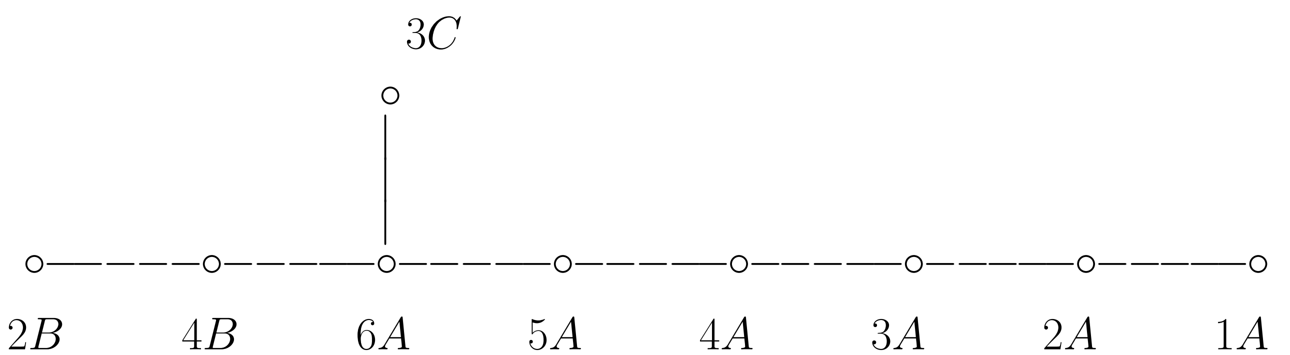
Given a VOA $V = \oplus_{i\geq 0} V_i$, the k-th product gives a bilinear mao $V_i \times V_j \longrightarrow V_{i+j-k-1}$. So, V_n under the n^{th} product is a finite dimensional algebra, denoted $(V_n, (n-1)^{th})$.

In addition, (a) if $dim(V_0) = 1$, $(V_1, 0^{th})$ is a Lie algebra; (b) if $dim(V_0) = 1$ and $dim(V_1) = 0$, then $(V_2, 1^{st})$ is a commutative algebra with a symmetric, associative form $(ab, c) = (a, bc)$. Algebras as in (b) are sometimes called *Griess algebras.*

There are many examples of algebras (b) with finite automorphism groups. The 196884-dimension algebra used to construct the Monster occurs this way in the Moonshine VOA $V^\natural$.

Now suppose $dim(V_0) = 1$ and that $e \in V_2$ is a conformal vector of central charge $\frac{1}{2}$ and that the subVOA generated by e is simple. Miyamoto showed that e gives $t_e \in Aut(V)$ of order 1 or 2 (called a *Miyamoto involution*).

In the special case of a dihedral VOA (generated by a pair of such conformal vectors e, f), the degree 2 algebra has integral forms. Those which are maximal integral forms and invariant under the dihedral group $\langle t_e, t_f \rangle$ were classified in the thesis of Greg Simon (U Michigan, 2016). For the nine types of dihedral VOAs (classified by Sakuma; they correspond to nodes of the *extended* E_8*-diagram*, displayed below), there is just one maximal invariant form in all cases but $2A$, in which case there are three.



(1)

This classification of maximal invariant forms in V_2 does not (yet) extend to invariant forms in the entire dihedral VOA.

6 Modular Moonshine of Borcherds and Ryba

Borcherds and Ryba wrote several articles about Modular Moonshine (positive characteristic) which imitated the story of the Monster and the graded representation $V^\natural$ and modular forms, but for smaller sporadic groups.

One of their cases is especially interesting. In $\mathbb{M}$, take g a $3C$-element; then $C(g) = \langle g \rangle \times S$, where $S \cong F_3$ a sporadic simple group of order $2^{15}3^{10}5^3 7^2 13 \cdot 19 \cdot 31$ (Thompson's group).

Borcherds and Ryba used K, Borcherds's $\mathbb{Z}[\frac{1}{2}]$-form in $V^\natural$, then took its 0th Tate cohomology group

$$\hat{H}^0(\langle g \rangle, K) := K^g/(1+g+g^2)K.$$

This inherits structure to make a VA over $\mathbb{F}_3$. It looked like the classical E_8 type VA over $\mathbb{F}_3$, but nonzero terms occur only in degrees 0, 3, 6 , . . . and have respective dimensions $1, 248, \ldots$, just like for the genuine E_8 VA in degrees 0, 1, 2, ... There was no obvious isomorphism (which would triple the degree of the grading) between these two VAs over $\mathbb{F}_3$.

To prove existence of an isomorphism, Lam and I adapted a "covering algebra" idea of Frohardt-Griess, which is illustrated in the following example.

Example 6.1. F algebraically closed field of characteristic 3. The Lie algebra $a_2(F)$ has a 1-dimensional central ideal, Z. While $Aut(a_2(F))$ is $PGL(3, F){:}2$, it turns out that $Aut(a_2(F)/Z) \cong G_2(F)$ (well known to experts in modular Lie algebras). Our proof is easier than earlier ones. One takes the Lie algebra $d := d_4(F)$ and graph automorphism γ of order 3, then considers

$$0 < (1 + \gamma + \gamma^2)d < d^\gamma < d, \quad \text{dimensions } 0,\ 7,\ 14,\ 28.$$

The group $G_2(F)$ acts on each subobject and on the 7-dimensional quotient Lie algebra $d^\gamma/(1 + \gamma + \gamma^2)d$. One can see inside d (look at the long roots) a copy of $a_2(F)$ which maps onto $d^\gamma/(1+\gamma+\gamma^2)d$; the image is isomorphic to $a_2(F)/Z$. Therefore $Aut(a_2(F)/Z)$ contains a copy of $G_2(F)$. This containment is equality.

Lam and I took M, the standard integral form for V_{E_8}, and a sublattice M' of K (integral form in $V^\natural$) which "covered" the Tate cohomology group $K^g/(1 + g + g^2)K$. (*Roughly,* M' is the $\mathbb{Z}$-span of the image of M under a map suggested by $x \mapsto x \otimes x \otimes x$ for $x \in EE_8 \cong \sqrt{2}E_8$; think of the containment of lattices $EE_8 \perp EE_8 \perp EE_8 \leq \Lambda$, the Leech lattice). This led to an isomorphism.

An application of the Borcherds-Ryba theory is a new proof that the group F_3 of Thompson embeds in $E_8(3)$ (first proof 1974, by Thompson and P. Smith, used a study of Dempwolff decompositions and computer work). This VA viewpoint gives a nontrivial homomorphism of $C(g)/\langle g\rangle \cong F_3$ into the group $E_8(3)$ without knowing anything about the structure of $C(g)$.

References. (see these articles for further references; my web site contains files of certain articles)

Griess, Robert L.; Lam, Ching Hung; Applications of vertex algebra covering procedures to Chevalley groups and modular moonshine, about 29 pages. arXiv:1308.2270; to appear in International Mathematics Research Notes.

MR2928458 Dong, Chongying; Griess, Robert L., Jr. Integral forms in vertex operator algebras which are invariant under finite groups. J. Algebra 365 (2012), 184198. 17B69; 17 January, 2012, http://arxiv.org/abs/1201.3411

Dong, Chongying; Griess, Robert L., Jr.; Lattice-integrality of certain group-invariant integral forms in vertex operator algebras, about 11 pages, arXiv:1405.4476 ; to appear in Journal of Algebra.

Daniel Frohardt and Robert L. Griess, Jr., Automorphisms of Modular Lie Algebras, Nova Journal of Algebra and Geometry, Vol. 1, No. 4, 1992, 339-345.

(pdf is on my web site)