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Two criteria for solvability of finite groups

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This is a joint paper with

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This paper is dedicated to the memory of our dear friend and colleague ZVI ARAD,

who passed away on February 4, 2018.

Introduction.

The aim of this paper is to prove the following two criteria for solvability of finite groups. Our first solvability criterion is the subject of the following theorem.

Theorem 1

Let G be a finite group of order n containing a subgroup A of prime power index p^r. Suppose that A contains a cyclic subgroup B satisfying $[A : B] \leq 2$. Then G is a solvable group.

Before stating the second criterion, we need two definitions and some preliminary remarks. If G is a finite group, then

$$\psi(G) = \sum_{x \in G} o(x),$$

where $o(x)$ denotes the order of x. Thus $\psi(G)$ is the sum of the orders of all elements of G.
Moreover, the cyclic group of order n is denoted by C_n. For example,

$$\psi(C_4) = 1 + 2 + 4 + 4 = 11$$

and

$$\psi(C_2 \times C_2) = 1 + 2 + 2 + 2 = 7.$$

In our previous paper we proved the following result (see [5], Theorem 1 and Corollary 4).

Theorem 2

Let G be a finite non-cyclic group of order n. Then

$$\psi(G) \le \frac{7}{11}\psi(C_n).$$

Moreover, if n is odd, then

$$\psi(G) < \frac{1}{2}\psi(C_n).$$

Thus the sum of element orders of C_n is by far bigger than that of any other group of order n.

Our second solvability criterion for groups G of order n refers to the ratio $\psi(G)/\psi(C_n)$ which is always ≤ 1. We proved:

Theorem 3

Let G be a finite group of order n and suppose that

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n).$$

Then G is a solvable group.

In particular, this theorem implies the following result.

Theorem 4

If G is a non-solvable group of order n, then

$$\psi(G) < \frac{1}{6.68}\psi(C_n).$$

In particular, this holds for all non-abelian simple groups.

We continue now with a series of remarks related to the above mentioned results. In these remarks G denotes a finite group.

- **Remark 1** We applied our first solvability criterion:

Theorem 1

Let G be a finite group of order n containing a subgroup A of prime power index p^r. Suppose that A contains a cyclic subgroup B satisfying $[A:B] \leq 2$. Then G is a solvable group.

in the proof of our second solvability criterion:

Theorem 3

Let G be a finite group of order n and suppose that

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n).$$

Then G is a solvable group.

- **Remark 2** For the proof of Theorem 1, we used the following two results.
- **Remark 3** Kegel-Wielandt's Theorem:

Theorem KW

If $G = AB$, where A and B are nilpotent subgroups of G, then G is solvable.

Wieland proved this result under the assumption that A and B are of co-prime orders (see [10]) and Kegel proved it in the general case, relying on Wielandt's result (see [7]).

- **Remark 4** Szep's conjecture, proved by Elsa Fisman and Zvi Arad:

Conjecture SFA

If $G = AB$, where A and B are subgroups of G with non-trivial centers, then G is not a non-abelian simple group.

(see the papers [8] and[9] of Szep, with the conjecture and partial results and see the paper [2] of Fisman and Arad, with the proof of the conjecture.)
The proof of Fisman and Arad is both simple and complicated.
The idea of the proof is simple. Suppose that

$$G = AB$$

and suppose that $a \in Z(A)$, $b \in Z(B)$ and $a, b \neq 1$. Then $G = C_G(a)C_G(b)$ and it is easy to see that this implies that

$$cl(a)cl(b) = cl(ab).$$

(Let g, h be arbitrary elements of G. Since $G = C_G(a)C_G(b)$, there exist $x \in C_G(a)$ and $y \in C_G(b)$ such that $xy = gh^{-1}$. Clearly

$$a^g b^h = (a^{gh^{-1}} b)^h = (a^{xy} b)^h = (a^y b^y)^h = (ab)^{yh}.$$

Thus $cl(a)cl(b) = cl(ab)$.)

Thus if $G = C_G(a)C_G(b)$, then a product of two non-trivial conjugacy classes of G is equal to a conjugacy class of G. Hence it is sufficient "only" to show that this is impossible in a finite non-abelian simple group. That is the complicated part of the proof.

- **Remark 5** The proof of the Kegel-Wieland's theorem does not rely upon the classification of finite simple groups, but the proof of Fisman and Arad does rely on it. Therefore our criteria for solvability also rely upon the classification of finite simple groups.
- **Remark 6** The assumptions of the Szep's conjecture do not imply solvability of G. For example, the non-solvable group

$$G = SL(2,5) \times C_2$$

is a product of two subgroups with non-trivial centers. But it is not simple.

- **Remark 7** For the proof of

Theorem 3

Let G be a finite group of order n and suppose that

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n).$$

Then G is a solvable group.

we also used the following five results

- **Remark 8** A result of H.Amiri, S.M.Jafarian Amiri and I.M.Isaacs (see Corollary B in [1]):

Theorem AAI

If R is a normal cyclic Sylow subgroup of G, then

$$\psi(G) \leq \psi(R)\psi(G/R),$$

with equality if and only if R is central in G.

- **Remark 9** Herstein's theorem (see [4]):

Theorem H

If G contains an abelian maximal subgroup, then G is solvable.

- **Remark 10** Our result (see Proposition 2.5 in [5]):

Theorem HLM

Let p be the maximal prime divisor of $|G|$ and suppose that $[G : \langle x\rangle] < 2p$ for some $x \in G$. Then either the Sylow p-subgroup of G is cyclic and normal in G or G is solvable.

In our proof we knew that G contains no cyclic normal Sylow subgroups. Hence the inequality $[G : \langle x\rangle] < 2p$ implied the solvability of G.

- **Remark 11** A theorem of Marshall Hall, Jr. (see Theorem 3.1 in [3]):

Theorem MH

Let p be a prime and let $n = 1 + rp$, with $1 < r < (p + 3)/2$. Then no group has n Sylow p-subgroups, unless either $n = q^t$ where q is a prime, or $r = (p - 3)/2$ and $p > 3$ is a Fermat prime.

We used this result for $p = 7$ and $n = 1 + 2 \cdot 7$. This value of n is impossible, since $1 + 2 \cdot 7 = 15$ is not a prime power and 7 is a Mersenne prime but not a Fermat prime, so the fact that $2 = (7 - 3)/2$ does not help either.

- **Remark 12** A Theorem of Andrea Lucchini (see Theorem 2.20 in [6]):

Theorem L

Let A be a cyclic proper subgroup of G and let $K = core_G(A)$. Then $[A : K] < [G : A]$, and in particular , if $|A| \geq [G : A]$, then $K > 1$.

- **Remark 13** The upper bound $\frac{7}{11}$ in

Theorem 2

Let G be a finite non-cyclic group of order n. Then

$$\psi(G) \leq \frac{7}{11}\psi(C_n).$$

is best possible. For example, as shown above, $\psi(C_2 \times C_2) = 7$ and $\psi(C_4) = 11$. Therefore

$$\psi(C_2 \times C_2) = \frac{7}{11}\psi(C_4).$$

But not only this group satisfies this equality. We have shown in [5] that if $n = 4k$, k odd, then the non-cyclic groups

$$G = C_{2k} \times C_2$$

of order n satisfy the equality $\psi(G) = \frac{7}{11}\psi(C_n)$.

- **Remark 14** Our solvability criterion in

Theorem 1

Let G be a finite group of order n containing a subgroup A of prime power index p^r. Suppose that A contains a cyclic subgroup B satisfying $[A : B] \leq 2$. Then G is a solvable group.

is quite delicate. The smallest non-abelian simple group A_5 contains a dihedral subgroup A of order 10, but $[A_5 : A] = 6$, not a prime power.

On the other hand, the simple group PSL(2,7) contains a subgroup A of index 8 and A contains a normal cyclic subgroup B of order 7, but $[A : B] = 3$, not 2.

- **Remark 15** Notice that $\psi(A_5) = 211$ and $\psi(C_{60}) = 1617$. Therefore

$$\psi(A_5) = \frac{211}{1617}\psi(C_{60}) > \frac{1}{7.67}\psi(C_{60}).$$

So our lower bound $\frac{1}{6.68}$ in

Theorem 3

Let G be a finite group of order n and suppose that

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n).$$

Then G is a solvable group.

is not very far from the best possible one. We could not replace $\frac{1}{6.68}$ by $\frac{1}{7.67}$.

As a matter of fact, we believe that the following conjecture holds:

Conjecture 5

If G is a group of order n and

$$\psi(G) > \frac{211}{1617}\psi(C_n),$$

then G is solvable.

If true, this lower bound is certainly best possible.
We also state a stronger version of Conjecture 5.

Conjecture 6

If G is a non-solvable group of order n, then

$$\psi(G) \leq \frac{211}{1617}\psi(C_n),$$

with equality if and only if $G = A_5$.

As mentioned above, the simple group A_5 satisfies the equality:

$$\psi(G)/\psi(C_n) = \frac{211}{1617} \sim \frac{1}{7.664},$$

but for the other five smallest finite non-abelian simple groups this ratio is lower than $\frac{1}{18}$.

Finally we mention our Proposition 2.6, which was used in the proof of Theorem 3:

Proposition 2.6

If H is a normal subgroup of G, then

$$\psi(G) \leq \psi(G/H)|H|^2.$$

We conjecture that the following "companion" inequality is also true:

Conjecture 7

If H is a subgroup of G, then

$$\psi(G) \leq \psi(H)(|G|/|H|)^2.$$

Our paper consists of four sections. Following the Introduction, the sections are: Preliminary results related to $\psi(G)$, Proof of Theorem 1 and Proof of Theorem 3. We continue with a sketch of the proofs

Preliminary results related to $\psi(G)$.

In [5], we proved the following lemma.

Lemma 2.1

Let n be a positive integer and suppose that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$$

where p_i are primes, $p_1 < p_2 < \cdots < p_r = p$ and α_i are positive integers. Then:

$$\psi(C_{p_i^{\alpha_i}}) = \frac{p_i^{2\alpha_i+1} + 1}{p_i + 1}$$

and

$$\psi(C_n) = \prod_{i=1}^{r} \psi(C_{p_i^{\alpha_i}}).$$

This implies the next lemma.

Lemma 2.2

Using the above notation we get

$$\psi(C_n) > \prod_{i=1}^{r} \frac{p_i}{p_i+1} n^2.$$

Proof:

$$\psi(C_{p_i^{\alpha_i}}) = \frac{p_i^{2\alpha_i+1}+1}{p_i+1} > \frac{p_i^{2\alpha_i+1}}{p_i+1} = (p_i^{\alpha_i})^2 \frac{p_i}{p_i+1}. \blacksquare$$

Using this inequality, we get quite easily the following results, which we shall use for the proof of Theorem 3.

Lemma 2.3

Using the above notation we get

1. If $1 \leq r \leq 4$ and $p_r = p > 7$, then
$$\psi(C_n) > \frac{385}{96} \cdot \frac{n^2}{p+1}.$$
2. If $r \geq 5$, then
$$\psi(C_n) > \frac{385}{96} \cdot \frac{n^2}{p+1}.$$
3. If $r = 4$ and $p_4 = 7$, then
$$\psi(C_n) > \frac{35}{96} n^2.$$

Proof: We shall prove case (3) only.

If $r = 4$ and $p_4 = 7$, then $p_1 = 2$, $p_2 = 3$ and $p_3 = 5$. Thus

$$\psi(C_n) > \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot n^2 = \frac{35}{96} n^2. \blacksquare$$

Proof of Theorem 1.

Theorem 1

Let G be a finite group of order n containing a subgroup A of prime power index p^r. Suppose that A contains a cyclic subgroup B satisfying $[A : B] \leq 2$. Then G is a solvable group.

Proof: We assume that

$$B < A < G,$$

with

$$B \text{ cyclic}, \ [A : B] \leq 2 \text{ and } [G : A] = p^r.$$

Let P be a Sylow p-subgroup of G. Then $G = PA$. Our proof is by induction on the order of G.

Using the theorems of Kegel and Wielandt and of Fisman and Arad we show that G cannot be a non-abelian simple group. So let N be a minimal normal subgroup of G. Since $G = PA$, it follows that

$$G/N = (PN)/N \cdot (AN/N).$$

Since $AN/N \cong A/(A \cap N)$, G/N satisfies our assumptions and by induction G/N is solvable. It remains only to show that N is solvable.

Since NA is a subgroup of G containing A and $G = PA$, it follows that $NA = (NA \cap P)A$. Thus NA satisfies the assumptions of our theorem.

If $NA < G$, then by the inductive hypothesis NA is solvable and hence N is solvable, as required.

Finally, if $NA = G$, then $G = PA$ implies that

$$|N||A|/|N \cap A| = |P||A|/|P \cap A|.$$

Thus $|N|/|A \cap N| = |P|/|A \cap P|$ and $N = (A \cap N)S$, where S is a Sylow p-subgroup of N. Hence N satisfies the assumptions of our theorem and by induction N is solvable. The proof of the theorem is complete. ■

Proof of Theorem 3.

Finally we are going to sketch the proof of Theorem 3.

Theorem 3

Let G be a finite group of order n and suppose that

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n).$$

Then G is a solvable group.

Proof: Suppose that G be a group of order n and

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n).$$

Our aim is to prove that G is solvable.
Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where p_i are primes, $p_1 < p_2 < \cdots < p_r = p$ and α_i are positive integers. Our proof is by induction on r.

If $r \leq 2$ then G is solvable, as required. So suppose that $r \geq 3$ and the theorem holds for groups of order which is a product of less than r distinct prime powers.
Suppose that G contains a normal cyclic Sylow subgroup R. Then by Theorem AAI and our assumptions, we have

$$\psi(R)\psi(G/R) \geq \psi(G) \geq \frac{1}{6.68}\psi(C_{|R|})\psi(C_{|G/R|}).$$

Since $\psi(R) = \psi(C_{|R|})$, it follows that $\psi(G/R) \geq \frac{1}{6.68}\psi(C_{|G/R|})$ and by our inductive hypothesis G/R is a solvable group. But then also G is solvable, as required. So we may assume that G has no normal cyclic Sylow subgroups.

Suppose, next, that H is an abelian subgroup of G with a prime index $[G:H]$. Then H is an abelian maximal subgroup of G and hence by Herstein's theorem G is solvable, as required. So we may assume that if H is an abelian subgroup of G, then the index $[G:H]$ is not a prime number. If either $p > 7$ or $r \geq 5$, then by Lemmas 2.3 we have

$$\psi(C_n) > \frac{385}{96} \cdot \frac{n^2}{p+1}.$$

Hence

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n) > \frac{1}{6.68} \cdot \frac{385}{96} \cdot \frac{n^2}{p+1} = \frac{385}{641.28} \cdot \frac{n^2}{p+1},$$

which implies that there exists $x \in G$ such that $|x| > \frac{385}{641.28} \cdot \frac{n}{p+1}$.

Hence

$$[G : \langle x \rangle] < \frac{641.28}{385}(p+1).$$

Since $r \geq 3$, it follows that $p \geq 5$ and $p + 1 \leq \frac{6}{5}p$.
Hence

$$[G : \langle x \rangle] < \frac{3847.68}{1925} p < 2p.$$

Since G has no normal cyclic Sylow subgroups, it follows by Theorem HLM that G is solvable, as required.

Therefore we may assume that $3 \leq r \leq 4$ and $p \leq 7$. This implies that we need to deal only with the following three cases: (i) $r = 4$ and $p = 7$, (ii) $r = 3$ and $p = 7$, and (iii) $r = 3$ and $p = 5$.

We shall sketch the proof in case (i) only. So assume that $r = 4$ and $p = 7$. Then $p_2 = 3$, $p_3 = 5$, $p_4 = 7$ and by Lemma 2.3(3) and we have

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n) > \frac{1}{6.68} \cdot \frac{35}{96} n^2 = \frac{35}{641.28} n^2,$$

which implies that there exists $x \in G$ such that $|x| > \frac{35}{641.28} n$. Hence

$$[G : \langle x \rangle] < \frac{641.28}{35} < 19.$$

If

$$7 \mid [G : \langle x \rangle],$$

then since an abelian subgroup cannot be of prime index in G, we must have

$$[G : \langle x \rangle] = 14.$$

Let $Q \leq \langle x \rangle$ be a cyclic Sylow 5-subgroup of G and let $N = N_G(Q)$. Then $N \geq \langle x \rangle$ and

$$14 = [G : N][N : \langle x \rangle] = (1 + 5k)[N : \langle x \rangle].$$

Since Q is a cyclic Sylow subgroup of G, $G = N$ is impossible, so $k > 0$ and we have reached a contradiction.
If

$$7 \nmid [G : \langle x \rangle],$$

let $P \leq \langle x \rangle$ be a cyclic Sylow 7-subgroup of G and let $N = N_G(P)$. Then $N \geq \langle x \rangle$ and

$$19 > [G : \langle x \rangle] = [G : N][N : \langle x \rangle] = (1 + 7k)[N : \langle x \rangle].$$

Again $k = 0$ is impossible and by Marshal Hall's theorem also $k = 2$ is impossible. Hence $k = 1$, which implies that

$$[G : \langle x \rangle] = 8[N : \langle x \rangle]$$

and $[N : \langle x \rangle] \leq 2$. Since $[G : N] = 8$, it follows by Theorem 1 that G is solvable, as required. The proof in case (i) is complete.
The other cases are similar, but case 3 is more complicated. ■

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The lecture is now complete.

THANK YOU for your ATTENTION!