A GAP-conjecture and its solution: Isomorphism classes of capable special p-groups of rank 2

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(joint with H. Heineken and R.F. Morse)

Definition 1. A group G is said to be capable if there exists a group H such that $G \cong H/Z(H)$, or equivalently, G is isomorphic to the inner automorphism group of a group H.

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Theorem 1. *Let A be a finitely generated abelian group written as*

$$A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus Z_{n_k}$$

such that $n_i \mid n_{i+1}$, where $\mathbb{Z}_n = \mathbb{Z}$, the infinite cyclic group, if $n = 0$. Then A is capable if and only if $k \geq 2$ and $n_{k-1} = n_k$.

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R. Baer, *Groups with preassigned central and central quotient groups*, Trans. Amer. Math. Soc. 44 (1938), 387-412.

F.R. Beyl, U. Felgner, and P. Schmid, *On groups occurring as center factor groups*, J. Algebra 61 (1979), 161-177.

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Definition 2. The epicenter $Z^*(G)$ of a group G is defined as

$$\bigcap \{\phi Z(E);\ (E, \phi) \text{ is a central extension of } G\}.$$

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Theorem 2. *A group is capable if and only* $Z^*(G) = 1$.

G. Ellis, *On the capability of groups*, Proc. Edinburgh Math Soc. 41 (1998), 487-495.

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Theorem 3. *$Z^*(G) = Z^\wedge(G) = \{a \in G \mid a \wedge g = 1_\wedge, \forall\, g \in G\}$, the exterior center of G.*

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A. Magidin and R.F. Morse, *Capable p-groups*, Proceedings Groups St. Andrews 2013, Lecture Notes LMS 422. (2015), 399-427.

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Theorem 4. *A special p-group of rank* 1 *(= extra special) is capable if and only if it is dihedral of order 8 or of order* p^3 *and exponent* p, $p > 2$.

H. Heineken, *Nilpotent groups of class 2 that can appear as central quotient groups*, Rend. Sem. Mat. Univ. Padova, 84 (1990), 241-248.

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Theorem 5. *Let G be a special p-group or rank 2 which is capable. Then*

$$p^5 \leq |G| \leq p^7.$$

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The case $p = 2$:

Theorem 6. *Let G be a capable special 2-group of rank 2. Then G has exponent 4 and there are three isomorphism classes, if $|G| = 2^5$ and 2^6, and one isomorphism class, if $|G| = 2^7$.*

From now on: $p > 2$.

GAP output: special p-groups of rank 2 and order p^5 for $2 < p \leq 37$:

	$\exp G = p$	
p	Total	Capable
3	1	1
5	1	1
7	1	1
11	1	1
13	1	1
17	1	1
19	1	1
23	1	1
29	1	1
31	1	1
37	1	1

GAP output: special p-groups of rank 2 and order p^5 for $2 < p \leq 37$:

	exp $G = p$		exp $G = p^2$	
p	Total	Capable	Total	Capable
3	1	1	10	3
5	1	1	12	3
7	1	1	14	3
11	1	1	18	3
13	1	1	20	3
17	1	1	24	3
19	1	1	26	3
23	1	1	30	3
29	1	1	36	3
31	1	1	38	3
37	1	1	44	3

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	exp $G = p$	
p	Total	Capable
3	3	3
5	3	3
7	3	3
11	3	3
13	3	3
17	3	3
19	3	3
23	3	3
29	3	3
31	3	3
37	3	3

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p	Total	Capable	Total	Capable
3	3	3	32	3
5	3	3	38	3
7	3	3	44	3
11	3	3	56	3
13	3	3	62	3
17	3	3	74	3
19	3	3	80	3
23	3	3	92	3
29	3	3	110	3
31	3	3	116	3
37	3	3	134	3

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	$\exp G = p$	
p	Total	Capable
3	2	1
5	2	1
7	2	1
11	2	1

GAP output: special p-groups of rank 2 and order p^7 for $2 < p \leq 11$:

	exp $G = p$		exp $G = p^2$	
p	Total	Capable	Total	Capable
3	2	1	97	1
5	2	1	136	1
7	2	1	184	1
11	2	1	298	1

Theorem 7. *Let G be a special p-group of rank 2, exponent p and order p^n, $5 \leq n \leq 7$. If G is capable, then there exists exactly one isomorphism class for $n = 5$ and 7, and three classes for $n = 6$.*

Theorem 7. *Let G be a special p-group of rank 2, exponent p and order p^n, $5 \leq n \leq 7$. If G is capable, then there exists exactly one isomorphism class for $n = 5$ and 7, and three classes for $n = 6$.*

A. Magidin, *On the capability of finite groups of class 2 and prime exponent*, Publ. Math. Debrecen, 85 (2014) 309-337.

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Consider $G = K \rtimes L$, $K = \langle a, b \mid a^{p^2} = b^{p^2} = 1,\ a^b = a^{p+1} \rangle$ and L an elementary abelian p-group of rank n.

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"**Proposition 1.**" *Let p be an odd prime. The groups defined by the following presentations contain all the capable special p-groups of rank 2 of order p^{4+n} with $G^p = G'$, exponent p^2 and $n \geq 1$:*

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$$\begin{aligned} G(m_1, \ldots, m_n) = \\ &\langle a, b, x_1, \ldots, x_n \mid a^{p^2} = b^{p^2} = x_1^p = \cdots = x_n^p = 1, \\ &a^b = a^{p+1}, a^{x_i} = a^{s_i p+1} b^{t_i p}, b^{x_i} = a^{u_i p} b^{-s_i p+1}, 1 \leq i \leq n \\ &[x_j, x_k] = 1,\ 1 \leq j < k \leq n \rangle, \end{aligned} \tag{1.1}$$

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$$G(m_1, \dots, m_n) = \langle a, b, x_1, \dots, x_n \mid a^{p^2} = b^{p^2} = x_1^p = \cdots = x_n^p = 1, \quad a^b = a^{p+1}, a^{x_i} = a^{s_i p+1} b^{t_i p}, b^{x_i} = a^{u_i p} b^{-s_i p+1}, 1 \leq i \leq n \quad [x_j, x_k] = 1,\ 1 \leq j < k \leq n \rangle, \tag{1.1}$$

where $0 \leq s_i, t_i, u_i < p$ and $m_i = \begin{pmatrix} s_i & t_i \\ u_i & -s_i \end{pmatrix}$ for $i = 1, \dots, n$.

The case exp $G = p^2$:
Consider $G = K \rtimes L$, $K = \langle a, b \mid a^{p^2} = b^{p^2} = 1,\ a^b = a^{p+1} \rangle$ and L an elementary abelian p-group of rank n.

Proposition 1. *Let p be an odd prime. The groups defined by the following presentations* **are all capable and in particular** *contain all the capable special p-groups of rank 2 of order p^{4+n} with $G^p = G'$, exponent p^2 and $n \geq 1$:*

$$G(m_1, \dots, m_n) = \langle a, b, x_1, \dots, x_n \mid a^{p^2} = b^{p^2} = x_1^p = \cdots = x_n^p = 1, a^b = a^{p+1}, a^{x_i} = a^{s_i p+1} b^{t_i p}, b^{x_i} = a^{u_i p} b^{-s_i p+1}, 1 \leq i \leq n\ [x_j, x_k] = 1,\ 1 \leq j < k \leq n \rangle, \tag{1.1}$$

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$$\mathcal{E}_1 = \{G(m) \mid 0 \neq \textit{ det m and -det m a quadratic residue mod p}\},$$

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$$\mathcal{E}_2 = \{G(m) \mid 0 \neq \text{ det } m, \text{ and -det } m \text{ a quadratic nonresidue mod } p\},$$

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$$\mathcal{E}_3 = \{G(m) \mid \text{ det } m = 0 \text{ and } m \neq \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}, u \in \mathbb{Z}_p\}.$$

Conjecture. For $A \in SL(2, p)$ we have $G(m) \cong G(m^A)$.

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No! If $m = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $m^A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. But $G(m) \ncong G(m^A)$.

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Proposition 2. *Let* $m = \begin{pmatrix} s & t \\ u & -s \end{pmatrix}$ *and* $k \in \mathbb{Z}_p^*$. *Then* $G(m) \cong G(km)$.

Let $m = \begin{pmatrix} s & t \\ u & -s \end{pmatrix}$ and $\bar{m} = \begin{pmatrix} \bar{s} & \bar{t} \\ \bar{u} & -\bar{s} \end{pmatrix}$. Set

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$$G(m) = \left\langle \begin{matrix} a, b, x; a^{p^2}, b^{p^2}, x^p, [a, b] = a^p, \\ [a, x] = a^{ps} b^{pt}, [b, x] = a^{up} b^{-sp} \end{matrix} \right\rangle$$

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and

$$G(\bar{m}) = \left\langle \begin{array}{l} \bar{a}, \bar{b}, \bar{x}; \bar{a}^{p^2}, \bar{b}^{p^2}, \bar{x}^p, [\bar{a}, \bar{b}] = \bar{a}^p, \\ [\bar{a}, \bar{x}] = \bar{a}^{p\bar{s}} \bar{b}^{p\bar{t}}, [\bar{b}, \bar{x}] = \bar{a}^{\bar{u}p} \bar{b}^{-\bar{s}p} \end{array} \right\rangle.$$

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Find α_1, β_1, γ_1, α_2, β_2, γ_2, γ with $\bar{a} = a^{\alpha_1} b^{\beta_1} x^{\gamma_1}$, $\bar{b} = a^{\alpha_2} b^{\beta_2} x^{\gamma_2}$, $\bar{x} = x^{\gamma}$ such that the relations of $G(\bar{m})$ are satisfied.

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Find α_1, β_1, γ_1, α_2, β_2, γ_2, γ with $\bar{a} = a^{\alpha_1} b^{\beta_1} x^{\gamma_1}$, $\bar{b} = a^{\alpha_2} b^{\beta_2} x^{\gamma_2}$, $\bar{x} = x^{\gamma}$ such that the relations of $G(\bar{m})$ are satisfied.

Remark. By Proposition 2 we can assume that $\gamma = 1$.

Proposition 3. *There exist* $\bar{a}, \bar{b}, \bar{x} \in G(m)$ *such that the relations* $[\bar{a}, \bar{x}] = \bar{a}^{p\bar{s}} \bar{b}^{p\bar{t}}$ *and* $[\bar{b}, \bar{x}] = \bar{a}^{p\bar{u}} \bar{b}^{-p\bar{s}}$ *are satisfied if and only if there exists*

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Remark. If $0 \neq \det\, m = \det\, \bar{m}$, then there exists $A \in SL(2, p)$ such that $m^A = \bar{m}$, or equivalently $mA = A\bar{m}$. (Note: $tr(m) = tr(\bar{m}) = 0$.)

Goal: For given α_1, α_2, β_1 β_2 find γ_1, γ_2 such that $[\bar{a}, \bar{b}] = \bar{a}^p$ is satisfied.

Observation: The relation $[\bar{a}, \bar{b}] = \bar{a}^p$ results into a 2×2 linear system of equations of the form $B \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$,

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(2) $G(m) \cong G\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$, *if $0 \neq$ det m and $-$det m is a quadratic residue mod p.*

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(3) $G(m) \cong G\left(\begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix}\right)$, *where r is a primitive root mod p, if $0 \neq$ det m and $-$det m is a quadratic nonresidue mod p.*

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(2) $G(m) \cong G\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$, *if* $0 \neq \det m$ *and* $-\det m$ *is a quadratic residue mod* p.

(3) $G(m) \cong G\left(\begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix}\right)$, *where* r *is a primitive root mod* p, *if* $0 \neq \det m$ *and* $-\det m$ *is a quadratic nonresidue mod* p.

Notation

$$[0] = \{G(m);\ \det(m) \equiv 0 \bmod p\},$$

$$[q] = \{G(m);\ 0 \not\equiv \det(m) \bmod p \text{ and } -\det(m) \text{ is a quadratic residue mod } p\},$$

$$[n] = \{G(m);\ 0 \not\equiv \det(m),\ -\det(m) \text{ a quadratic nonresidue mod } p\}.$$

The case $|G| = p^7$ and $\exp(G) = p^2$.

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Theorem 10. *Let* $\mathcal{M}_p = \left\{ \begin{pmatrix} s & t \\ u & -s \end{pmatrix} ; s, t, u \in \mathbb{Z}_p \right\}$*, where p is an odd prime. Any three linearly independent matrices* $m_1, m_2, m_3 \in \mathcal{M}_p$ *determine a capable special p-group of rank 2, order* p^7 *and exponent* p^2*. Any two such groups are isomorphic.*

Proposition 11. *Consider* $G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)$ *and* $G(m_1, m_2, m_3)$ *with*

Proposition 11. *Consider* $G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)$ *and* $G(m_1, m_2, m_3)$ *with*

$$m_1 = s\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + t\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + u\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

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$$m_3 = s''\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + t''\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + u''\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

Proposition 11. *Consider* $G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)$ *and* $G(m_1, m_2, m_3)$ *with*

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with

$$\det\begin{pmatrix}s & t & u\\ s' & t' & u'\\ s'' & t'' & u''\end{pmatrix} \not\equiv 0 \mod p,$$

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$$m_1 = s\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + t\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + u\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$
$$m_2 = s'\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + t'\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + u'\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$
$$m_3 = s''\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + t''\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + u''\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

with

$$\det\begin{pmatrix}s & t & u\\ s' & t' & u'\\ s'' & t'' & u''\end{pmatrix} \not\equiv 0 \mod p,$$

then

$$G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right) \cong G(m_1, m_2, m_3).$$

Theorem 12. *Let p be an odd prime. Then there exist at least three isomorphism classes of capable special p-groups of rank 2, order p^6 and exponent p^2.*

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Conjecture 13. Let p be an odd prime. Then there are at most three isomorphism classes of capable special p-groups of rank 2, order p^6 and exponent p^2.

Proposition 14. *The groups* $G\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)$,

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$G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ -v & 0\end{pmatrix}\right)$*, where* v *is a quadratic nonresidue mod* p*, and*

Proposition 14. *The groups* $G\left(\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\right)$,
$G\left(\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},\begin{pmatrix}0 & 1\\-v & 0\end{pmatrix}\right)$, *where* v *is a quadratic nonresidue mod* p, *and* $G\left(\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\right)$ *are pairwise nonisomorphic.*

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$G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)$ *are pairwise nonisomorphic.*

Sketch of proof. Isomorphism types of maximal subgroups which are capable and special of rank 2, order p^5 and exponent p^2

Proposition 14. *The groups* $G\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)$, $G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ -v & 0\end{pmatrix}\right)$, *where* v *is a quadratic nonresidue mod* p, *and* $G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)$ *are pairwise nonisomorphic.*

Sketch of proof. Isomorphism types of maximal subgroups which are capable and special of rank 2, order p^5 and exponent p^2

	[0]	[q]	[n]
$G\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)$	Yes	Yes	Yes
$G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ -v & 0\end{pmatrix}\right)$	No	Yes	Yes
$G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)$	Yes	Yes	No

Conjecture 15. Let $G(m, m')$ be a capable special p-group of rank 2 of order p^6 and exponent p^2. Then $G(m, m')$ is isomorphic to

Conjecture 15. Let $G(m, m')$ be a capable special p-group of rank 2 of order p^6 and exponent p^2. Then $G(m, m')$ is isomorphic to $G\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)$, $G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)$, or $G\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)$,