Almost Engel compact groups

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Joint work with Pavel Shumyatsky

Engel groups

Notation: left-normed simple commutators

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Recall: a group G is an Engel group if for every $x, g \in G$,

$$[x, g, g, \dots, g] = 1,$$

where g is repeated sufficiently many times depending on x and g.

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Recall: a group G is an Engel group if for every $x, g \in G$,

$$[x, g, g, \dots, g] = 1,$$

where g is repeated sufficiently many times depending on x and g.

Clearly, any locally nilpotent group is an Engel group.

Known facts on finite groups

Zorn's Theorem

A finite Engel group is nilpotent.

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Baer's Theorem

If g is an Engel element of a finite group G,
that is, $[x, g, \ldots, g] = 1$ for every $x \in G$, then $g \in F(G)$.

Here, $F(G)$ is the Fitting subgroup, largest normal nilpotent subgroup.

Engel compact groups

J. Wilson and E. Zelmanov, 1992

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If a Lie algebra L satisfies a nontrivial identity and is generated by d elements such that each commutator in these generators is ad-nilpotent, then L is nilpotent.

Yu. Medvedev, 2003

Any Engel compact (Hausdorff) group is locally nilpotent.

Almost Engel groups

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Definition

A group G is almost Engel if for every $g \in G$ there is a finite set $\mathscr{E}(g)$ such that for every $x \in G$,

$$[x, \underbrace{g, g, \ldots, g}_{n}] \in \mathscr{E}(g) \qquad \text{for all } n \geqslant n(x, g).$$

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Theorem 1 (Almost Engel $\Rightarrow$ almost locally nilpotent)

Suppose that G is an almost Engel compact (Hausdorff) group. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

(...there is also a locally nilpotent subgroup of finite index: $C_G(N)$.)

Three parts of the proof

1. **Finite groups**, a quantitative version.
2. **Profinite groups:** using finite groups, Wilson–Zelmanov theorem.
3. **Compact groups:** reduction to profinite case
   using structure theorems for compact groups.

Some notation

If G is an almost Engel group, then for every $g \in G$ there is a unique minimal finite set $\mathscr{E}(g)$ with the property that for every $x \in G$,

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The nilpotent residual of a group G is

$$\gamma_\infty(G) = \bigcap_i \gamma_i(G),$$

where $\gamma_i(G)$ are terms of the lower central series ($\gamma_1(G) = G$, and $\gamma_{i+1}(G) = [\gamma_i(G), G]$).

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Suppose that G is a finite group and there is a positive integer m such that $|\mathscr{E}(g)| \leqslant m$ for every $g \in G$. Then $|\gamma_\infty(G)|$ is bounded in terms of m.

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(...And G also has a nilpotent normal subgroup of bounded index: $C_G(\gamma_\infty(G))$.)

About the proof for finite groups

Lemma

In any almost Engel group G, the Engel sink is the set

$$\mathscr{E}(g) = \{z \in G \mid z = [z, g, \ldots, g]\}$$

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Lemma

In a finite group, if A is an abelian section, acted on by g of <u>coprime</u> order, then $[A, g] = \{[a, g, \dots, g] \mid a \in A\}$ for any number of g, so $[A, g] \subseteq \mathscr{E}(g)$.

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Proof: $C_{[A,g]}(g) = 1 \quad \Rightarrow \quad [A, g] = \{[b, g] \mid b \in [A, g]\}$. □

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If $|\mathscr{E}(g)| \leqslant m$ for all $g \in G$, then $G/F(G)$ is of m-bounded exponent.

Proof: Clearly, g centralizes its powers. Hence for any $z \in \mathscr{E}(g^k)$ we have

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By Baer's theorem, then $g^{m!} \in F(G)$, so $G/F(G)$ has exponent dividing $m!$. □

Further proof for finite groups

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Proof of Theorem 1 (that $|\gamma_\infty(G)|$ is m-bounded)

is by induction on $|G/F(G)|$...

Profinite groups

Recall:

Inverse limits of finite groups.

Topological groups. Quotients only by closed subgroups.

Open subgroups have finite index and are also closed.

Sylow theory. Pronilpotent (=pro-(finite nilpotent)) groups are Cartesian products of pro-p groups.

Largest normal pronilpotent subgroup (closed).

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Proof: For any $h \in H$ there is a normal subgroup R such that $\mathscr{E}(h) \cap R = \{1\}$ with nilpotent H/R.

Then $\mathscr{E}(h) \subseteq R$, so in fact $\mathscr{E}(h) = \{1\}$,

so h is an Engel element. □

Bounded version for profinite groups

Theorem 2 on finite groups immediately implies the following.

Corollary

Suppose that G is an almost Engel profinite group and there is a positive integer m such that $|\mathscr{E}(g)| \leqslant m$ for every $g \in G$.

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Suppose that G is an almost Engel profinite group and there is a positive integer m such that $|\mathscr{E}(g)| \leqslant m$ for every $g \in G$.
Then G has a finite normal subgroup N of order bounded in terms of m such that G/N is locally nilpotent.

General case of profinite groups

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First goal: a pronilpotent normal subgroup of finite index.

In the proof, a certain section is considered, and the Baire category theorem is applied.

A piece of proof

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In an almost Engel profinite group G, the sets

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Then equations (*) show that for every $n \in N$ the sink $\mathscr{E}(yn)$ has an element in every coset $z_i N$,

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Thus, $|G/F(G)|$ is finite, where $F(G)$ is the largest pronilpotent normal subgroup (which is also locally nilpotent by Lemma above). Further arguments are by induction on $|G/F(G)|$ and are similar to those for finite groups.

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Note that a simple compact Lie group is a linear group.

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An almost Engel divisible group is an Engel group.

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Hence $Z(G_0) = G_0$ is abelian by the structure theorem.

Using the profinite case

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... etc., in the end use Theorem 3 on profinite again.

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So far, the case of finite groups has been done:

Theorem 4

Suppose that G is a finite group and there is a positive integer r such that $\langle \mathscr{E}(g) \rangle$ has rank at most r for every $g \in G$. Then the rank of $\gamma_\infty(G)$ is bounded in terms of r.