Infinite – dimensional Leibniz algebras in the spirit of infinite group theory

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Many concepts in group theory have analogues in theory of algebras (associative and non – associative). For example, a natural analogue of the notion of a subgroup is a subalgebra, the concept of a normal subgroup is an ideal, the notion of a subnormal subgroup is a subideal, and so on. The notions of the center, the upper and lower central series, the concepts of nilpotency, solvability, and so on are introduced for algebras as it was done in group theory. There are some problems in group theory, which have analogs in theory of algebras. In particular, in the theory of Lie algebras, there is a large part, in which the analogs of many problems of group theory were considered. I. Stewart called it "Infinite - dimensional Lie algebras in the spirit of the infinite group theory". Perhaps it's not just analogies, we are talking about problems, approaches, setting tasks, because the final result was not always completely analogous to the group theory. This part of the theory of Lie algebras was developed very intensively, there is a huge array of articles and several books. Since in the Lie algebras this approach has turned out to be successful and fruitful, it is natural to apply it also to the generalizations of Lie algebras. One of such generalizations are Leibniz algebras.

Let L be an algebra over a field F with the binary operations + and [,]. Then L is called a ***Leibniz algebra*** (more precisely a ***left Leibniz algebra***) if it satisfies the Leibniz identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$$

for all a, b, c $\in$ A.

Leibniz algebras appeared first in the papers of A.M. Bloh

BA1965. BLOH A.M. *On a generalization of the concept of Lie algebra.* Doklady AN USSR – **165** (1965), 471 – 473.

BA1967. BLOH A.M. *Cartan – Eilenberg homology theory for a generalized class of Lie algebras.* Doklady AN USSR – **175** (1967), 824 – 826.

BA1971. BLOH A.M. *A certain generalization of the concept of Lie algebra.* Algebra and number theory. Moskov. Gos. Pedagogical Inst., Uchenye Zapiski – **375** (1971), 9 – 20.

A.M. Bloh used the term D – algebras in these papers. However, in that time these works were not in demand, and they have not been properly developed. Only after two decades, a real interest to Leibniz algebras arose. It was happened thanks to the work of J.L. Loday

LJ1993. LODAY J.L. *Une version non commutative des algebres de Lie; les algebras de Leibniz.* Enseign. Math. **39** (1993), 269 – 293.

J.L. Loday "rediscovered" these algebras and used the term ***Leibniz algebras*** since it was Leibniz who discovered and proved the *Leibniz rule* for differentiation of functions.

An algebra R over a field F is called ***right Leibniz*** if it satisfies the Leibniz identity

$$[a, [b, c]] = [[a, b], c]] – [[a, c], b]$$

for all a, b, c $\in$ A.

Note at once that the classes of left Leibniz algebras and right Leibniz algebras are different. The following simple example justifies it.

Let F be an arbitrary field and L be a vector space over F having a basis $\{a, b\}$. Define the operation $[,]$ on L by the following rule:

$$[a, a] = b, [b, a] = [b, b] = 0, [a, b] = b.$$

It is not hard to check that L becomes a left Leibniz algebra. But

$$0 = [[a, a], a] \neq [[a, a], a] + [a, [a, a]] = [a, b] = b.$$

Let R be a right Leibniz algebra, then put $\Subset a, b \Supset = [b, a]$. Then we have

$$\Subset\Subset a, b \Supset, c \Supset = [c, [b, a]] = [[c, b], a] - [[c, a], b] =$$
$$\Subset a, \Subset b, c \Supset - \Subset b, \Subset a, c \Supset\Supset.$$

Thus, this substitution leads us to a left Leibniz algebra. Similarly, we can make a transfer from a left Leibniz algebra to a right Leibniz algebra.

An algebra L over a field F is called a ***symmetric Leibniz algebra*** if it is both a left and right Leibniz algebra.

We prefer to work with left Leibniz algebras eventhough many authors prefer to consider right Leibniz algebras. Thus further the term a Leibniz algebra stands for a left Leibniz algebra.

The Leibniz algebras appeared to be naturally related to several areas such as differential geometry, homological algebra, classical algebraic topology, algebraic K – theory, loop spaces, noncommutative geometry, and so on. They found some applications in physics.

Note at once that if L is a Lie algebra, then

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

It follows that

$$[[a, b], c] = -[[b, c], a] - [[c, a], b] = [a, [b, c]] + [b, [c, a]] = [a, [b, c]] - [b, [a, c]],$$

which shows that every Lie algebra is a Leibniz algebra.

Conversely, suppose that $[a, a] = 0$ for each element $a \in L$. Then for arbitrary elements $a, b \in L$ we have

$$0 = [a + b, a + b] = [a, a] + [a, b] + [b, a] + [b, b] = [a, b] + [b, a].$$

It follows that $[a, b] = -[b, a]$. Then we obtain

$$0 = [[a, b], c] - [a, [b, c]] + [b, [a, c]] = [[a, b], c] + [[b, c], a] - [[a, c], b] =$$
$$[[a, b], c] + [[b, c], a] + [[c, a], b]$$

for all $a, b, c \in A$. Thus, Lie algebras can be characterized as Leibniz algebras in which $[a, a] = 0$ for every element a. Or Lie algebras can be characterized as anticommutative Leibniz algebras.

Figuratively speaking the situation with Leibniz and Lie algebras is similar to those one that we have in non – abelian and abelian groups.

The theory of Leibniz algebras has been developing quite intensively but un-even. On the one hand, some analogues of important results from the theory of Lie algebras were proven. On the other hand, some natural questions about the structure of Leibniz algebras are not considered. For example, until very recently, even cyclic subalgebras of Leibniz algebras were not fully described. Almost all authors considered finite – dimensional Leibniz algebras only. Moreover, they typically studied finite – dimensional Leibniz algebras over a field of characteristic 0, mostly over $\mathbb{R}$ or $\mathbb{C}$. Even a description of Leibniz algebras, having dimension 3, is done for the case of characteristic 0 only. This is similar to the situation that was in the initial period of the formation of group theory, when group theory began to develop solitary as the theory of finite groups.

Here we want to show only some recent results about the generalized nilpotent Leibniz algebras, which are the analogs of some famous classical results of group theory.

A Leibniz algebra L is called ***abelian*** (or trivial) if $[a, b] = 0$ for every elements $a, b \in L$. In particular, an abelian Leibniz algebra is a Lie algebra.

Denote by **Leib**(L) the subspace, generated by the elements $[a, a], a \in L$. It is not hard to prove that **Leib**(L) is an ideal of L and $L/$**Leib**(L) is a Lie algebra. Conversely, if H is an ideal of L such that L/H is a Lie algebra, then **Leib**(L) $\leq H$.

The ideal **Leib**(L) is called the ***Leibniz kernel*** of algebra L.

We note the following important property of the Leibniz kernel:

$$[[a, a], x] = [a, [a, x]] – [a, [a, x]] = 0.$$

This property shows that **Leib**(L) is an abelian subalgebra of L.

Let L be a Leibniz algebra. Define the lower central series of L

$$L = \gamma_1(L) \geq \gamma_2(L) \geq \ldots \geq \gamma_\alpha(L) \geq \gamma_{\alpha+1}(L) \geq \ldots \gamma_\delta(L)$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \cap_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ. The last term $\gamma_\delta(L)$ is called the ***lower hypocenter*** of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, \ldots, L] \ldots]$ is the ***left normed commutator*** of k copies of L.

We remark that if A, B are ideals of a Leibniz algebra L, then, in general, [A, B] needs not be an ideal. A corresponding example was constructed in

BD2013. BARNES D. *Schunck Classes of soluble Leibniz algebras.* Communications in Algebra, 41(2013), 4046–4065

However if H is an ideal then $\gamma_j(H)$ is an ideal of L for each positive integer j.
Since an operation in Leibniz algebra is not anticommutative, we must say about the left and right center.

The ***left*** (respectively ***right***) ***center*** $\zeta^{left}(L)$ (respectively $\zeta^{right}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{left}(L) = \{ x \in L \mid [x, y] = 0 \text{ for each element } y \in L \}$$

(respectively,

$$\zeta^{right}(L) = \{ x \in L \mid [y, x] = 0 \text{ for each element } y \in L \}).$$

It is not hard to prove that the left center of L is an ideal, but it is not true for the right center. Moreover, **Leib**$(L) \leq \zeta^{\mathbf{left}}(L)$, so that $L/\zeta^{\mathbf{left}}(L)$ is a Lie algebra. The right center is an subalgebra of L, and in general, the left and right centers are different; they even may have different dimensions. The following examples shows it.

EXAMPLE 1. Let F be a field. Put $L = Fe_1 \oplus Fe_2 \oplus Fe_3 \oplus Fe_4$ and define an operation [,] by the following rule:

$$[e_1, e_1] = e_2,\ [e_1, e_2] = -e_2 - e_3,\ [e_1, e_3] = e_2 + e_3,\ [e_1, e_4] = 0,$$
$$[e_2, e_1] = 0,\ [e_3, e_1] = 0,\ [e_4, e_1] = e_2 + e_3,$$
$$[e_j, e_k] = 0 \text{ for all } j, k \in \{2, 3, 4\}.$$

It is possible to check that this operation defines a Leibniz algebra. We can see that $\zeta^{\mathbf{right}}(L) = Fe_4$ and $\zeta^{\mathbf{right}}(L)$ is not an ideal. Furthermore, $\zeta^{\mathbf{left}}(L) = Fe_2 \oplus Fe_3$, so that $\zeta^{\mathbf{right}}(L) \cap \zeta^{\mathbf{left}}(L) = \langle 0 \rangle$, $\mathbf{dim}_F(\zeta^{\mathbf{right}}(L)) = 1$, $\mathbf{dim}_F(\zeta^{\mathbf{left}}(L)) = 2$. Note also that $[L, L] =$ **Leib**$(L) = \zeta^{\mathbf{left}}(L)$.

EXAMPLE 2. Let F be a field. Put $L = Fe_1 \oplus Fe_2 \oplus Z$ where a subspace Z has a countable basis $\{z_n, \mid n \in \mathbf{N}\}$. Put $[z_n, x] = 0$ for every $x \in L$ and

$$[e_1, e_1] = [e_2, e_2] = [e_1, e_2] = [e_2, e_1] = z_1,\ [e_1, z_1] = [e_2, z_1] = 0.$$

By such definitions, we have

$$0 = [[e_j, e_k], e_m] \text{ and } [e_j, [e_k, e_m]] - [e_k, [e_j, e_m]]\ 0 - 0 = 0 \text{ for all } j, k, m \in \{1, 2\}.$$

Take into account the equalities

$$0 = [[e_1, e_2], z] = [e_1, [e_2, z]] - [e_2, [e_1, z]],\ 0 = [[e_2, e_1], z] = [e_2, [e_1, z]] - [e_1, [e_2, z]],$$

we obtain $[e_2, [e_1, z]] - [e_1, [e_2, z]]$. Now we put

$$[e_1, z_j] = z_j,\ [e_2, z_j] = z_{j+1} \text{ for all } j > 1.$$

By this definition, we have

$$0 = [[e_j, z], e_k] \text{ and } [e_j, [z, e_k]] - [z, [e_j, e_k]] = [e_j, 0] - 0 = 0,$$
$$0 = [[z, e_j], e_k] \text{ and } [z, [e_j, e_k]] - [e_j, [z, e_k]] = 0 - [e_j, 0] = 0 \text{ for all } j, k \in \{1, 2\}.$$

As we have seen above

$$[[e_j, e_k], z] = [e_j, [e_k, z]] - [e_k, [e_j, z]] \text{ for all } j, k \in \{1, 2\}.$$

Hence, L is a Leibniz algebra. By it construction Z is a left center of L, the right center coincides with the center and coincides with Fz_1, so that, the left center has finite codimension (and therefore, infinite dimension) and the right center and the center have finite dimension. By the construction, $[L, L] = Z$. Furthermore

$$[e_1 + z_1, e_1 + z_1] = [e_1, e_1] + [z_1, e_1] + [e_1, z_1] + [z_1, z_1] = z_1,$$
$$[e_1 + z_j, e_1 + z_j] = [e_1, e_1] + [z_j, e_1] + [e_1, z_j] + [z_j, z_j] = z_1 + z_j \text{ for } j > 1.$$

It follows that **Leib**$(L) = Z$.

These both examples were constructed in a paper

KOP2016. KURDACHENKO L.A., OTAL J., PYPKA A.A. *Relationships between factors of canonical central series of Leibniz algebras.* European Journal of Mathematics – 2016, 2, 565 – 577.

The ***center*** $\zeta(L)$ of L is defined by the rule:

$$\zeta(L) = \{ x \in L \mid [x, y] = 0 = [y, x] \text{ for each element } y \in L \}.$$

The center is an ideal of L. In particular, we can consider the factor – algebra $L/\zeta(L)$.

Now we define the upper central series

$$<0> = \zeta_0(L) \leq \zeta_1(L) \leq \zeta_2(L) \leq \ldots \leq \zeta_\alpha(L) \leq \zeta_{\alpha+1}(L) \leq \ldots \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \cup_{\mu<\lambda} \zeta_\mu(L)$ for the limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_\infty(L)$ of this series is called the ***upper hypercenter*** of L.

A Leibniz algebra L is said to be ***hypercentral*** if it coincides with the upper hypercenter. Denote by **zl**(L) the length of upper central series of L.

The introduced here concepts of the upper and lower central series for Leibniz algebras are an analogous of others similar concepts, which became standard in several algebraic structures. They play an important role, for example, in Lie algebras and group theory. Following this analogy, we say that a Leibniz algebra L is called ***nilpotent***, if there exists a positive integer k such that $\gamma_k(L) = <0>$. More precisely, L is said to be ***nilpotent of nilpotency class c*** if $\gamma_{\mathbf{c}+1}(L) = <0>$, but $\gamma_{\mathbf{c}}(L) \neq <0>$. We denote the nilpotency class of L by **ncl**(L).

It is a well – known that in nilpotent Lie algebras and nilpotent groups the lower and the upper central series have the same length.

Consider now the factors $\gamma_k(L)/\gamma_{k+1}(L)$, $k \in \mathbf{N}$. By definition $[L, \gamma_k(L)] = \gamma_{k+1}(L)$. It is not hard to show that $[\gamma_k(L), L] = [\gamma_k(L), \gamma_1(L)] \leq \gamma_{k+1}(L)$.

Let

$$<0> = C_0 \leq C_1 \leq \ . \ . \ . \leq C_\alpha \leq C_{\alpha+1} \leq . \ . \ . \ C_\gamma = L$$

be an ascending series of ideals of Leibniz algebra L. This series is called ***central*** if $C_{\alpha+1}/C_\alpha \leq \zeta(L/C_\alpha)$ for each ordinal $\alpha < \gamma$. In other words, $[C_{\alpha+1}, L], [L, C_{\alpha+1}] \leq C_\alpha$ for each ordinal $\alpha < \gamma$.

We note the following properties of central series.

PROPOSITION 1. *Let L be an Leibniz algebra over a field F, and*

$$<0> = C_0 \leq C_1 \leq \ . \ . \ . \leq C_n = L$$

be a finite central series of L. Then

(i) $\gamma_j(L) \leq C_{n-j+1}$, so that $\gamma_{n+1}(L) = <0>$.
(ii) $C_j \leq \zeta_j(L)$, so that $\zeta_n(L) = L$.
(iii) If j, k are positive integer such that $k \geq j$, then $[\gamma_j(L), \zeta_k(L)], [\zeta_k(L), \gamma_j(L)] \leq \zeta_{k-j}(L)$.

COROLLARY. *Let L be an Leibniz algebra over a field F and suppose that L has a finite central series*

$$<0> = C_0 \leq C_1 \leq \ . \ . \ . \leq C_n = L.$$

Then L is nilpotent and ***ncl***$(L) \leq n$. *Furthermore, the upper central series of L is finite,* $\zeta_\infty(L) = L$, ***zl***$(L) \leq n$. *Moreover,* ***ncl***$(L) =$ ***zl***(L).

These results have been obtained in the paper

KOP2016. KURDACHENKO L.A., OTAL J., PYPKA A.A. *Relationships between factors of canonical central series of Leibniz algebras.* European Journal of Mathematics – 2016, 2, 565 – 577.

The last Corollary shows that a Leibniz algebra L is nilpotent if and only if there is a positive integer k such that $L = \zeta_k(L)$. The least positive integer having this property coincides with nilpotency class of L. So, as in the cases of Lie algebras and groups, the definition of nilpotency can be given here using the notion of the upper central series.

Here it will be appropriate to note the fact that the Leibniz algebra L can be associative. Indeed, if $[L, L] = \gamma_2(L) \leq \zeta(L)$, then $0 = [[x, y], z] = [x, [y, z]]$ for all $x, y, z \in L$. Conversely, suppose that L is associative. Then, taking into account the equility $[[x, y], z] = [x, [y, z]]$, from $[[x, y], z] = [x, [y, z] – [y, [x, z]]$ we derive that $[y, [x, z]] = 0$. Since it is true for all $x, y, z \in L$, $[L, L] \leq \zeta^{\mathbf{right}}(L)$. Furthermore, $0 = [y, [x, z]] = [[y, x], z]$, which shows that $[L, L] \leq \zeta^{\mathbf{left}}(L)$. So we obtain

PROPOSITION 2. *Let L be a Leibniz algebra over a field F. Then L is associative if and only if $[L, L] \leq \zeta(L)$.*

The concepts of upper and lower central series introduced here immediately leads us to the following classes of Leibniz algebras.

A Leibniz algebra L is said to be ***hypercentral*** if it coincides with the upper hypercenter.

A Leibniz algebra L is said to be ***hypocentral*** if its lower hyporcenter is trivial.

In the case of finite dimensional algebras, these two concepts coincide, but in general, these two classes are very different.

Thus, for finitely generated hypercentral Leibniz algebras we have

THEOREM 1. *Let L be a finitely generated Leibniz algebra over a field F. If L is hypercentral, then L is nilpotent. Moreover, L has finite dimension. In particular, a finitely generated nilpotent Leibniz algebra has finite dimension.*

This result is an analog of a similar group theoretical result proved by A. I. Mal'cev

MA1949. MALTSEV A.I. *Nilpotent torsion – free groups.* Izvestiya AN USSR, series math. – **13**(1949), no. 3, 201 – 212.

At the same time, a finitely generated hypocentral Leibniz algebra can have infinite dimension.

Let A be a Leibniz algebra over a field F and d be an element of A. Put

$$\mathbf{ln}_1(d) = d,\ \mathbf{ln}_2(d) = [d, d],\ \mathbf{ln}_{k+1}(d) = [d, \mathbf{ln}_k(d)],\ k \in \mathbf{N}.$$

These elements are called the ***left normed commutators of the element d***.

LEMMA 1. *Let L be a Leibniz algebra over a field F, $a \in L$. Then every non – zero commutator of k copies of an element a with any bracketing is coincides with $\mathbf{ln}_k(a)$. Hence a cyclic subalgebra $\langle a \rangle$ is generated as a subspace by the elements $\mathbf{ln}_k(a)$, $k \in \mathbf{N}$.*

If the elements $d_j = \mathbf{ln}_j(d)$, $j \in \mathbf{N}$ are linearly independent, then a cyclic algebra $D = \langle d \rangle$ has the lower central series

$$D = \gamma_1(D) \geq \gamma_2(D) \geq \ldots \geq \gamma_j(D) \geq \gamma_{j+1}(D) \geq \ldots \langle 0 \rangle$$

of the length ω, and $\gamma_j(D) = \oplus_{t \geq j} Fd_t$, $j \in \mathbf{N}$. In this case, we will say that an ***element d has infinite depth.***

Thus, a cyclic Leibniz algebra < d > where an element a has infinite depth is hypocentral and has infinite dimension. At the same time, D has a trivial center.

A Leibniz algebra L is said to be ***locally nilpotent*** if every finite subset of L generates a nilpotent subalgebra.

That is why, hypercentral Leibniz algebras give us examples of locally nilpotent algebras.

We obtained the following characterization of hypercentral Leibniz algebras.

THEOREM 2. *Let L be a Leibniz algebra over a field F. Then L is hypercentral if and only if for each element* $a \in L$ *and every countable subset* $\{ x_n \mid n \in \mathbf{N} \}$ *of elements of L there exists a positive integer k such that all commutators* $[x_1, \ldots, x_j, a, x_{j+1}, \ldots, x_k]$ *are zeros for all* j, $0 \leq j \leq k$.

COROLLARY. *Let L be a Leibniz algebra over a field F. Then L is hypercentral if and only if every subalgebra of L having finite or countable dimension is hypercentral.*

These results are analogs of some group-theoretical results of S.N. Chernikov.

Let L be a Leibniz algebra. If A, B are nilpotent ideals of L, then their sum A + B is a nilpotent ideal of L. This result has been proved in a paper

BD2013. BARNES D. *Schunck classes of soluble Leibniz algebras.* Communications in Algebra, 41(2013), 4046 – 4065.

In this connection, the following question arises: Whether an analogous assertion is valid for locally nilpotent ideals? For Lie algebras this assertion takes place, as it was shown by B. Hartley in the paper

HB1967. HARTLEY B. *Locally nilpotent ideals of a Lie algebras.* Proc. Cambridge Phil. Society, 63(1967), 257 – 272.

Our next result gives an affirmative answer to this question.

THEOREM 3. *Let L be a Leibniz algebra over a field F, A, B be locally nilpotent ideals of L. Then A + B is locally nilpotent.*

COROLLARY 1. *Let L be a Leibniz algebra over a field F and $\mathfrak{S}$ be a family of locally nilpotent ideals of L. Then a subalgebra generated by $\mathfrak{S}$ is locally nilpotent.*

COROLLARY 2. *Let L be a Leibniz algebra over a field F. Then L has the greatest locally nilpotent ideal.*

Let L be a Leibniz algebra over field F. The greatest locally nilpotent ideal of L is called the ***locally nilpotent radical*** of L and will be denoted by **Ln**(L).

These results are analogues of the results in groups proved by K.A. Hirsch

HK1955. HIRSCH K.A. *Über local – nilpotente Gruppen,* Math. Z. – **63** (1955), 290 – 291.

and B.I. Plotkin

PB1955. PLOTKIN B.I. *Radical groups.* Math. sbornik, **37** (1955), 507 – 526.
PB1958. PLOTKIN B.I. *Generalized soluble and generalized nilpotent groups.* Uspekhi mat. nauk, **13** (1958), no. 4, 89 – 172.

The subalgebra **Nil**(L) generated by all nilpotent ideals of L is called the ***nil radical*** of L. If L = **Nil**(L), then L is called a Leibniz ***nil – algebra***. Every nilpotent Leibniz algebra is a nil – algebra, but converse is not true even for a Lie algebra. Every Leibniz nil – algebra is locally nilpotent, but converse is not true even for a Lie algebra. Moreover, there exists a Lie nil – algebra, which is not hypercentral. There is a corresponding example in Chapter 6 of the book

AS1974. AMAYO R.K., STEWART I. *Infinite dimensional Lie algebras.* **Noordhoff Intern. Publ.**: Leyden, 1974.

Note the following important properties of locally nilpotent Leibniz algebras.

THEOREM 4. *Let L be a locally nilpotent Leibniz algebra over a field F.*

(i) If A, B, $A \leq B$ are the ideals of L such that factor B/A is L – chief, then B/A is central in L (that is $B/A \leq \zeta(L/A)$). In particular, $\mathbf{dim}_F(B/A) = 1$.

(ii) If A is a maximal subalgebra of L, then A is an ideals of L.

Let L be a Leibniz algebra over the field F and H a subalgebra of L. The ***idealizer*** of H is defined by the following rule:

$$\mathbf{I}_L(H) = \{ x \in L \mid [h, x], [x, h] \in H \text{ for all } h \in H \}.$$

It is possible to prove that the idealizer of H is a subalgebra of L. If L is a hypercentral (in particular, nilpotent) Leibniz algebra, then $H \neq \mathbf{I}_L(H)$. This leads us to the following class of Leibniz algebras.

Let L be a Leibniz algebra over field F. We say that L ***satisfies the idealizer condition*** if $\mathbf{I}_L(A) \neq A$ for every proper subalgebra A of L.

A subalgebra A is called ***ascendant*** in L, if there is an ascending chain of subalgebras

$$A = A_0 \leq A_1 \leq \ldots A_\alpha \leq A_{\alpha+1} \leq \ldots A_\gamma = L$$

such that A_α is an ideal of $A_{\alpha+1}$ for all $\alpha < \gamma$.

It is possible to prove that L satisfies the idealizer condition if and only if every subalgebra of L is ascendant. The last our result is the following

THEOREM 5. *Let L be a Leibniz algebra over a field F. If L satisfies the idealizer condition then L is locally nilpotent.*

This result is analogous to the following result proved by B.I. Plotkin for groups.

PB1951. PLOTKIN B.I. *To the theory of locally nilpotent groups.* Doklary AN USSR **76** (1951), 655 – 657.

Again, it should be noted that Leibniz algebras with the idealizer condition will form a proper subclass of the class of locally nilpotent Leibniz algebras, since this is already the case for Lie algebras. A corresponding example could be found in Chapter 6 of the book

AS1974. AMAYO R.K., STEWART I. *Infinite dimensional Lie algebras.* **Noordhoff Intern. Publ.**: Leyden, 1974.

Grazie mille!