Regular Direct Limits of Symmetric Groups

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Joint work with Otto H. Kegel

19 -24 March 2018

Dedicated to the memories of
Prof. Michio Suzuki
Prof. Klaus Doerk,
Prof. Valeria Fedri

Hall's Universal group

A locally finite group U satisfying,
(i) Every finite group can be embedded into U,
(ii) Any two isomorphic finite subgroups of U are conjugate in U is called a universal group.

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P. Hall, Some constructions for locally finite groups J. London Math. Soc. **34**, (1959), 305-319.

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Construction of the group is in the following way:

Hall's Universal group

Let $G_1 = Sym(n_1)$ be a symmetric group on n_1 letters where $n_1 \geq 3$. Embed G_1 by right regular representation ρ_1 into $G_2 = Sym(G_1)$. Then embed G_2 by right regular representation ρ_2 into $G_3 = Sym(G_2)$ and continue like this. Then we obtain a direct system

$$G_1 \xrightarrow{\rho_1} G_2 \xrightarrow{\rho_2} G_3 \ldots$$

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Then the direct limit group

$$U = \lim_{i \to \infty} G_i \cong \bigcup_{i=1}^{\infty} G_i$$

is Hall's universal group.

Hall's Universal group

The properties of U are the followings:

- U contains isomorphic copy of every finite group.
- Any two isomorphic finite subgroups are conjugate in U.
- U is a simple, non-linear, locally finite group.

Hall's Universal group

- Let C_m denote the set of all elements of order $m > 1$ of U. Then C_m is a single class of conjugate elements and $U = C_m C_m$. In particular U is simple.

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- Every countably infinite locally finite group can be embedded into U.

Injectivity property

Injectivity: G is a universal group and if $H \leq K$, and K is finite with $\phi : H \to G$ is an injection, then ϕ can be extended to an injection $K \to G$.

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Since every finite group can be embedded in an alternating group, general linear group, a special linear group, by this property, we may write Hall's universal group as a direct limit of alternating groups.

Then the question of whether Hall's universal group can be written as a direct limit of other families of finite simple groups is answered by F. Leinen in [1].

F. Leinen, *Lokale systeme in universellen gruppen,* Arch. Math. **41**, (1983), pp. 401–403.

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$$\{PSL(n_i, \mathbf{F}_q)\}, \{PSU(n_i, \mathbf{F}_q)\}, \{PSp(2n_i, \mathbf{F}_q)\},$$
$$\{P\Omega^+(2n_i, \mathbf{F}_q)\}, \{P\Omega(2n_i + 1, \mathbf{F}_q)\},$$
$$\{P\Omega^-(2n_i + 2, \mathbf{F}_q)\}.$$

It is natural to ask whether U can be expressed as a union (direct limit) of the infinite simple locally finite groups $PSL(n_i, \overline{\mathbf{F}}_p)$, where $\overline{\mathbf{F}}_p$ is the algebraic closure of the field $\mathbf{F}_p$ with p elements. It is proved in [1, Theorem 1] together with A. Zalesski that the answer is positive.

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M. Kuzucuoğlu, A. E. Zalesskii; *Hall universal group as a direct limit of algebraic groups*, J. Algebra **192**, 55-60, (1997).

Theorem 1

Let p be any fixed prime. Hall's universal group is a direct limit of some groups $PSL(n_i, \overline{\mathbf{F}}_p)$, $i = 1, 2, \ldots$ such that all the sequent embeddings are rational maps (morphisms of algebraic groups).

Hickin's Theorem

Perhaps, one of the most striking property of Hall's universal group, in contrast to Sylow theory for finite groups was discovered by Hickin who proved in (1986), [1, Theorem 4]:

K. Hickin; *Universal locally finite central extensions of groups,* Proc. London Math. Soc. (3), **52**, 53–72,(1986).

Theorem 2

For every prime p, every countably infinite locally finite p-group can be embedded into U as a maximal p-subgroup.

Maximal p-subgroups in U

Surprisingly it follows from the above theorem that, U has isomorphic copy of countably infinite elementary abelian p-group and locally cyclic group C_{p^∞} as maximal p-subgroups.

So conjugacy of Sylow Theorems are not true for Hall's universal group.

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Could it be possible to have a maximal p-subgroup in U which is a maximal subgroup of U?
M. D. Molle in [1] proved that the answer is positive.

M. D. Molle; *Sylow subgroups which are maximal in the universal locally finite group of Philip Hall*, J. Algebra, **215**, 229–234, (1999).

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Theorem 3

The countable universal locally finite group U contains, for each prime p, a maximal subgroup that is a p-group.

Centralizers in Hall's Universal group

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Brian Hartley, *Simple locally finite groups. Finite and locally finite groups*, İstanbul, (1994), 1-44, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **471**, Kluwer Acad. Publ., Dordrecht, 1995.

Centralizers in Hall's Universal group

Theorem 4 (B. Hartley)

(a) *If F is a finite subgroup of U with trivial center, then $C_U(F)$ is isomorphic to U.*
(b) *If A is a finite abelian subgroup of U, then $C_U(A)/A$ is an infinite simple group.*

Centralizers in Hall's Universal group

Is it possible to find the structure of centralizers of centreless finite subgroups in U by using basic group theory?

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The answer is yes. Namely we prove the following:

Centralizers in Hall's Universal group

Theorem 5

Let U be the Hall's universal group and F be a finite centreless subgroup of U.
Then the centralizer $C_U(F)$ is isomorphic to U.
Moreover, as every finite group F is contained in a centerless finite subgroup B, we have
$U \cong C_U(B) \leq C_U(F)$.
Hence centralizer of every finite subgroup F of U contains an isomorphic copy of U.

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Otto H. Kegel, Mahmut Kuzucuoğlu; *Centralizers of finite subgroups in Hall's universal group*, Rend. Sem. Mat. Univ. Padova. **(138)**, (2017) 283–288.

Centralizer of right regular representation

We use the following basic results.

Lemma 6

$C_{Sym(G)}(r(G)) = l(G)$ and $C_{Sym(G)}(l(G)) = r(G)$.

$l(g_1)r(g_2) = r(g_2)l(g_1)$ if and only if
$xl(g_1)r(g_2) = xr(g_2)l(g_1)$ for any $x \in G$ if and only if
$(g_1^{-1}x)g_2 = g_1^{-1}(xg_2)$, for any $x \in G$.
So

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So

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Lemma 7

$l(G) \cap r(G) = l(Z(G)) = r(Z(G)) \cong Z(G)$.

Locally Finite Groups

Question What can be said about the group of automorphisms of a universal group?

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Page 185 in

Kegel O. H., Wehrfritz B. A. F., *Locally Finite Groups*, North-Holland Publishing Company - Amsterdam, 1973.

Locally Finite Groups

The following Theorem is proved:

Theorem 8

$Aut(U)$ is complete (i.e. $Aut(U)$ has no center and no outer automorphisms).

Theorem 9

$Inn(U)$ is the locally finite radical of $Aut(U)$.
(i.e. it is the largest locally finite normal subgroup of $Aut(U)$).

Paolini, Gianluca; Shelah, Saharon; The automorphism group of Hall's universal group. Proc. Amer. Math. Soc. **146** (2018), no. 4, 1439–1445.

Definition of κ-existentially closed groups

Let κ be an infinite cardinal. A group G is called a κ-existentially closed group if $|G| \geq \kappa$ and every consistent system of less than κ-many equations and in-equations with coefficients from G which has a solution in $H \geq G$, then they have a solution in G.

κ-existentially closed groups

Equivalent definition of κ existentially closed group for uncountable groups.

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Let κ be an uncountable cardinal. A group G of cardinality $|G| \geq \kappa$ is called κ-existentially closed if

- G contains an isomorphic copy of every group of cardinality less than κ, and
- every isomorphism between two subgroups of G of cardinality less than κ is induced by an inner automorphism of G.

κ-existentially closed groups

Hall's universal group is a locally finite group, so it does not contain elements of infinite order.

κ-existentially closed groups

But if we start with an infinite symmetric group and embed it by right regular representation into its symmetric group and continue with transfinite induction sufficiently large cardinal number of times and on the limit ordinals we take the union of the preceeding obtained subgroups, we obtain a direct limit group.

κ-existentially closed groups

Some of these groups have some interesting properties. O. H. Kegel will mention some of their basic properties. Moreover we determine the the structure of centralizers of some subgroups in these groups. They have similar properties as in the case of Hall's universal groups.

κ-existentially closed groups

Theorem 10

Let κ be a limit cardinal of cofinality $|I|$. Let G be the direct limit of symmetric groups G_i obtained by right regular representation of G_i into G_{i+1}, where $i \in I$ and for limit ordinals we take the union. Let F be a subgroup of G contained in G_i for some $i \in I$ with $Z(F) = 1$. Then the centralizer $C_G(F)$ is isomorphic to G.

κ-existentially closed groups

Corollary 11

Let G be the κ-existentially group of order inaccessible cardinal κ and F be any proper subgroup of G with $Z(F) = 1$. Then $C_G(F)$ is isomorphic to G.

Otto H. Kegel and Mahmut Kuzucuğlu, *κ-existentially closed groups*, J. Algebra, (2018) **499**, 298-310.

THANK YOU