Semi-extraspecial p-groups with an abelian subgroup of maximal possible order

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Introduction

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Note that extraspecial groups are trivially semiextraspecial.

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2. G/G' and $Z(G)$ are elementary abelian.

3. $|G : G'|$ is a square.

4. $|G'| \leq \sqrt{|G : G'|}$.

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Because of the Heisenberg groups, we know that for every prime p and positive integer a, there exist an ultraspecial group of order p^{3a}.

Isoclinism

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Definition: Two groups G and H are *isoclinic* if there exist isomorphisms $\alpha : G/Z(G) \to H/Z(H)$ and $\beta : G' \to H'$ such that $[\alpha(g_1 Z(G)), \alpha(g_2 Z(G))] = \beta([g_1, g_2])$ for all $g_1, g_2 \in G$.

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Using the Universal Coefficients Theorem, one can prove that if p is an odd prime, then every semiextraspecial p-group is isoclinic to a unique (up to isomorphism) semiextraspecial p-group of exponent p.

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Heineken and Verardi have provided examples of ultraspecial groups:

(1) No abelian subgroups of order p^{2a}. (2) One abelian subgroup of order p^{2a}.

Verardi has proved that if an ultraspecial group G of order p^{3a} has at least 2 abelian subgroups of order p^{2a}, then the number of abelian subgroups of order p^{2a} has the form $1 + p^h$ for some integer h that satisfies $0 \leq h \leq a$ and if $h > 0$, then h divides a.

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Verardi has proved when p is odd that if G is an ultraspecial group of order p^{3a} and exponent p and has $1 + p^a$ abelian subgroups of order p^{2a}, then G is isomorphic to the Heisenberg group of degree a.

Finite semifields

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We say $(F, +, *)$ is a *pre-semifield* if $(F, +)$ is an abelian group with at least two elements whose identity is 0 and $*$ is a multiplication that satisfies the distributive laws and $a * c = 0$ for $a, c \in F$ implies that $a = 0$ or $c = 0$.

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We say F is a *semifield* if in addition, F has an identity which we would denote by 1.

Semifield groups

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Let $(F, +, *)$ be a finite pre-semifield. We define the group $G(F)$ to be the group with the set $\{(a, b, c) \mid a, b, c \in F\}$.

We define the multiplication on $G(F)$ by

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1 * b_2).$$

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The multiplication we gave for $G(F)$ could be viewed as the multiplication in 3×3-matrices with 1's on the diagonal and entries in the semifield F.

Note that the sets
$A_1 = \{(a, 0, c) \mid a, c \in F\}$ and $A_2 = \{(0, b, c) \mid b, c \in F\}$
are abelian subgroups of $G(F)$ of order $|F|^2$.

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Theorem 1.

Let G be an ultraspecial group of order p^{3a} with two abelian subgroups of order p^{2a} that have exponent p. Then there is a semifield F of order p^a so that G is isomorphic to $G(F)$.

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An independent proof was given by Verardi in 1987 for odd p.

Isomorphic semifield groups

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The question that arises is: when do different (pre)-semifields give isomorphic semifield groups?

Definition: We say two (pre)-semifields F_1 and F_2 are *isotopic* if there exist isomorphic linear transformations $\alpha, \beta, \gamma : F_1 \to F_2$ such that $\gamma(a *_1 b) = \alpha(a) *_2 \beta(b)$ for all $a, b \in F_1$.

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It known that every semifield of order p, p^2, or 8 is isotopic to a field, and when p is an odd prime and $a \geq 3$ and when $p = 2$ and $a \geq 4$ there is some semifield of order p^a that is not isotopic to a field.

If F is a semifield, then we define F^{op} by $a *_{\mathrm{op}} b = b * a$ for all $a, b \in F$.

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We say that two semifields F_1 and F_2 are *anti-isotopic* if F_1 and F_2^{op} are isotopic.

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Theorem 2.

If F_1 and F_2 are semifields such that $G(F_1)$ and $G(F_2)$ are isomorphic, then F_1 and F_2 are either isotopic or anti-isotopic.

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Counting semifield groups

$\|F\|$	# isotopism classes	$\|G(F)\|$	# $G(F)$
2^3	1	2^9	1
2^4	3	2^{12}	3
2^5	6	2^{15}	4
2^6	332	2^{18}	184
3^3	2	3^9	2
3^4	27	3^{12}	19
3^5	23	3^{15}	15
5^3	4	5^9	?
7^3	?	7^9	?
7^4	356	7^{12}	227

Table: Number of semifield groups

One abelian subgroup of order p^{2n}

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We will start with a semifield F and we will need $\beta : F \times F \to F$ to be a biadditive map.

Define $\overline{\beta}(a, b) = \beta(a, b) - \beta(b, a)$.

Theorem 3.

Let F be a semifield of order p^n and let $\beta : F \times F \to F$ be a biadditive map. Consider the set $G = F \times F \times F$ and define a multiplication on G by

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1 * b_2 + \beta(b_1, b_2)).$$

Then the following hold:

- *G is a semi-extraspecial group with $G' = Z(G) = \{(0, 0, c) \mid c \in W\}$.*
- *$A = \{(a, 0, c) \mid a, c \in F\}$ is an abelian subgroup of order p^{2n}.*
- *$B = \{(0, b, c) \mid b, c \in F\}$ is a subgroup of order p^{2n}.*

Theorem (Theorem 3 continued).

- $G = AB$ *and* $A \cap B = G'$.
- *For an element* $v \in F \setminus \{0\}$, *we have* $\overline{\beta}(u, v) = 0$ *for all* $u \in F$ *if and only if* $(0, v, c) \in Z(B)$ *for all* $c \in F$. *In particular,* $B = C_G(0, v, c)$ *if and only if* $\overline{\beta}(u, v) = 0$ *for all* $u \in F$.
- $\overline{\beta}(u, v) = 0$ *for all* $u, v \in F$ *if and only if* B *is abelian.*

Theorem (Theorem 3 continued).

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- $\overline{\beta}(u, v) = 0$ for all $u, v \in F$ if and only if B is abelian.

We will write $G(F, \beta)$ for the group G in Theorem 3.

Theorem 4.

Let F be a semifield, let β be a biadditive map on F, and let $G = G(\alpha, \beta)$. Then $G(F) \cong G(F, \beta)$ if and only if there exists an additive map $f : F \to F$ so that $\overline{\beta}(v_1, v_2) = f(v_2) * v_1 - f(v_1) * v_2$ for all $v_1, v_2 \in F$.

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When V is an elementary abelian p-group of order p^n, it is not difficult to see that $|\mathrm{add}(V)| = (p^n)^n = p^{(n^2)}$.

Recall that a biadditive map $\gamma : F \times F \to F$ is *alternating* if $\gamma(v, v) = 0$ for all $v \in F$.

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Recall that if β is any biadditive map, then $\overline{\beta} \in \mathrm{alt}(F)$.

Using pointwise addition, we see that $\mathrm{alt}(F)$ is a group.

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Suppose that F is a semifield.

For each $f \in \mathrm{add}(V)$, we define $\varphi(f) : F \times F \to F$ by $\varphi(f)(v_1, v_2) = f(v_1) * v_2 - f(v_2) * v_1$.

Observe that $\varphi(f) \in \mathrm{alt}(F)$, so $\varphi : \mathrm{add}(F) \to \mathrm{alt}(F)$.

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In particular, $\varphi(\mathrm{add}(F))$ is a subgroup of $\mathrm{alt}(F)$.

Restating Theorem 4, if F is a semifield and β is a bilinear map on F, then $G(F, \beta) \cong G(F)$ if and only if $\overline{\beta} \in \varphi(\mathrm{add}(F))$.

Theorem 5.

If F is a semifield of order p^n and β is a bilinear map on F so that $\overline{\beta} \notin \varphi(\mathrm{add}(F))$, then $G(F, \beta)$ has exactly one abelian subgroup of order p^{2n}.

Theorem 5.

If F is a semifield of order p^n and β is a bilinear map on F so that $\overline{\beta} \notin \varphi(\mathrm{add}(F))$, then $G(F, \beta)$ has exactly one abelian subgroup of order p^{2n}.

Corollary 6.

For every prime p and for every integer $n \geq 3$, there exists an ultraspecial group G where $|G| = p^{3n}$ and G has an unique abelian subgroup A of order p^{2n}.

Theorem 7.

If G is an ultraspecial group of order p^{3n} and exponent p with an abelian subgroup of order p^{2n}, then there exists a semifield F and a biadditive map β so that $G \cong G(F, \beta)$.

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Question: If F_1 and F_2 are semifields with associated biadditive maps β_1 and β_2, then when are $G(F_1, \beta_1)$ and $G(F_2, \beta_2)$ isomorphic?

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If $G(F_1, \beta_1) \cong G(F_2, \beta_2)$, then F_1 and F_2 are isotopic.

(Note that anti-isotopic does not seems to work here.)

If $F_1 = F_2$ and $\overline{\beta}_2 \in \overline{\beta_1} + \varphi(\mathrm{Add}(F))$, then $G(F_1, \beta_1) \cong G(F_2, \beta_2)$.

Questions?