Virtually torsion-free covers of minimax groups (with an application to random walks)

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We will call such groups $\mathfrak{M}$*-groups*.

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Corollary

(Derek Robinson, 1975) *Every finitely generated, virtually solvable group of finite abelian section rank is an* $\mathfrak{M}$*-group.*

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• Finitely generated, virtually torsion-free $\mathfrak{M}$-groups have solvable word problem (Frank Cannonito and Derek Robinson, 1984).

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Conjecture

(1980s or earlier) This is possible if the group is finitely generated.

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Then G cannot be expressed as a quotient of a virtually torsion-free $\mathfrak{M}$-group.

Random walks on $\mathfrak{M}$-groups

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Theorem

(C. Pittet and L. Saloff-Coste, 2003) *Let G be a virtually torsion-free $\mathfrak{M}$-group with a finite symmetric generating set S. Then*

$$P_{(G,S)}(2t) \gtrsim \exp(-t^{\frac{1}{3}}).$$

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Conjecture

(*circa* 2006) *The virtually torsion-free hypothesis can be dropped from the above theorem.*

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Theorem

Let G be a finitely generated $\mathfrak{M}$-group, and write $N = \mathrm{Fitt}(G)$ and $S = \mathrm{solv}(G)$. Then there is a virtually torsion-free $\mathfrak{M}$-group G^* and an epimorphism $\phi : G^* \to G$ satisfying the following four properties, where $N^* = \mathrm{Fitt}(G^*)$ and $S^* = \mathrm{solv}(G^*)$.

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Let G be an $\mathfrak{M}$-group with a finite symmetric generating set S. If G is not virtually nilpotent, then

$$P_{(G,S)}(2t) \sim \exp(-t^{\frac{1}{3}}).$$

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Let π be a set of primes. If G is a group and A a $\mathbb{Z}G$-module, then we say that the action of G on A is *π-integral* if, for each $g \in G$, there are integers $\alpha_0, \alpha_1, \ldots, \alpha_m$ such that α_m is a nonzero π-number and $(\alpha_0 + \alpha_1 g + \cdots + \alpha_m g^m) \in \mathrm{Ann}_{\mathbb{Z}G}(A)$.

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Theorem

Let π be a set of primes and G an $\mathfrak{M}$-group. Write $N = \mathrm{Fitt}(G)$ and $Q = G/N$. Then the following two statements are equivalent.

(i) *G can be expressed as a homomorphic image of a virtually torsion-free $\mathfrak{M}_\pi$-group.*

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Theorem

Let π be a set of primes and G an $\mathfrak{M}$-group. Write $N = \mathrm{Fitt}(G)$ and $Q = G/N$. Then the following two statements are equivalent.

(i) *G can be expressed as a homomorphic image of a virtually torsion-free $\mathfrak{M}_\pi$-group.*

(ii) *Q is finitely generated, $\mathrm{spec}(N) \subseteq \pi$, and Q acts π-integrally on N_{ab}.*

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N_p is a locally compact topological group.

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We can find a closed radicable normal nilpotent subgroup R_0 of $G_{(p)}$ and a closed virtually torsion-free subgroup X such that $G_{(p)} = R_0 X$.

We can also form an inverse system

$$\cdots \to R_2 \to R_1 \to R_0$$

consisting of epimorphisms of radicable nilpotent X-groups such that each R_i contains a copy of P and the induced maps

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So R^* is torsion-free.

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Thanks for your attention!