On the nilpotency of the verbal subgroup corresponding to the Engel word

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joint work with
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Ischia Group Theory 2018
March 20th

Preliminaries

A group-word $w = w(x_1, \ldots, x_k)$ is a nontrivial element of the free group $F = F(x_1, \ldots, x_k)$ on free generators $x_1, \ldots, x_k$.

If G is a group, we think of w as a map $w : G^k \to G$.

We denote by $w(G)$ the verbal subgroup of G corresponding to the word w, that is the subgroup of G generated by the set

$$G_w = \{w(g_1, \ldots, g_k) \mid g_i \in G\}$$

of all w-values in G.

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Lower Central Words

Given an integer $k \geq 1$, the word $\gamma_k = \gamma_k(x_1, \ldots, x_k)$ is defined inductively by the formulae

$$\gamma_1 = x_1, \qquad \text{and} \qquad \gamma_k = [\gamma_{k-1}, x_k] = [x_1, \ldots, x_k] \quad \text{for } k \geq 2.$$

The subgroup of a group G generated by all γ_k-values is denoted by $\gamma_k(G)$, and this is the familiar kth term of the lower central series of G.

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Given an integer $k \geq 1$, the kth Engel word $\varepsilon_k = \varepsilon_k(x, y)$ is defined by the formulae

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A group-word is said to be a commutator word if it belongs to the commutator subgroup F'.

Among all the commutator words, there are multilinear commutator words, that is words obtained nesting commutators but using always different variables like for example

$$[[x_1, [x_2, x_3]], [x_4, x_5]].$$

For $k \geq 2$:

- γ_k is multilinear;
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Question

Can we grasp any information about $w(G)$ imposing some condition on G_w?

In the class of finite groups, one can think to impose some conditions on the orders of w-values.

Definition

A group G satisfies $\mathcal{P}(w)$ if $|ab| = |a||b|$ whenever a and b are w-values of coprime orders.

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Theorem (B. Baumslag, J. Wiegold - 2014)

Let G be a finite group and let $w = x$. If G satisfies $\mathcal{P}(w)$ then G is nilpotent.

Theorem (R. Bastos, P. Shumyatsky - 2016)

Let G be a finite group and let $w = [x, y]$. If G satisfies $\mathcal{P}(w)$ then G' is nilpotent.

$$w(G) = \begin{cases} G & \text{if} \quad w = x \\ G' & \text{if} \quad w = [x, y] \end{cases}$$

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Main Question

Question

Let w be a group-word and let G be a finite group satisfying $\mathcal{P}(w)$. Is then the verbal subgroup $w(G)$ nilpotent?

For noncommutator words the answer is NEGATIVE.

Counter-example (Noncommutator Word)

Choose a nonabelian finite simple group, say of exponent e, and the word x^n, where n is a divisor of e such that e/n is prime.

Even for commutator words the answer is NEGATIVE!

Counter-example (Commutator Word)

Consider the word $u = u(x, y) = [x, y^{10}, y^{10}, y^{10}]$ in the group $\mathbb{A}_5$: then all the u-values are either trivial or product of two 2-cycles.

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We suspect that the answer is positive in the case of multilinear commutator words.

Indeed, we answer in the POSITIVE when $w = \gamma_k$.

Theorem (R. Bastos, C. M., P. Shumyatsky - 2017)

Let G be a finite group and let $w = \gamma_k$. G satisfies $\mathcal{P}(w)$ if and only if $\gamma_k(G)$ is nilpotent.

Open Question

What about the nilpotency of the kth term $G^{(k)}$ of the derived series?

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What about the nilpotency of the kth term $G^{(k)}$ of the derived series?

For any $n, k \geq 1$, let v be the word defined as

$$v = v(x_1, \dots, x_k, y) = \varepsilon_n(\gamma_k, y) = [x_1, \dots x_k,_n y].$$

Theorem (C. M., A. Tortora)

If G is a finite group satisfying $\mathcal{P}(v)$, then $v(G)$ is nilpotent.

Notice that if $k = 1$, then $v = \varepsilon_n$ and so we answered in the positive for Engel words.

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Theorem (E. Detomi, M. Morigi and P. Shumyatsky - 2017)

Suppose that w is a multilinear commutator word. For any $n \geq 1$ the word $[w,_n y]$ is concise in residually finite groups.

A group-word w is said to be concise in a class of group $\mathcal{X}$ if whenever G_w is finite for a group $G \in \mathcal{X}$, it follows that $w(G)$ is finite.

Recall that a group G is said to be residually finite if for every $x \in G \setminus \{1\}$ there exists a normal subgroup N of G such that $x \notin N$ and G/N is finite.

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Theorem

If G is a residually finite group satisfying $\mathcal{P}(v)$ such that G_v is finite, then $v(G)$ is nilpotent.

- Being v concise and G_v finite, $v(G)$ is finite.
- Since G is residually finite, there exists a normal subgroup N of G such that $N \cap v(G) = 1$ and G/N is finite.
- $|xN| = |x|$ for any x v-value, and so $v(G/N)$ is nilpotent.
- $v(G/N) \simeq v(G)N/N \simeq v(G)/N \cap v(G) \simeq v(G)$.

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BIBLIOGRAPHY

B. Baumslag, J. Wiegold
A Sufficient Condition for Nilpotency in a Finite Group
preprint available at arXiv:1411.2877v1 [math.GR]

R. Bastos, P. Shumyatsky
A Sufficient Condition for Nilpotency of the Commutator Subgroup,
Siberian Mathematical Journal, **57** (2016), 762–763

R. Bastos, C. Monetta and P. Shumyatsky,
A Criterion for Metanilpotency of a Finite Group
preprint available at arXiv:1706.03133 [math.GR]

E. Detomi, M. Morigi and P. Shumyatsky
Words of Engel type are Concise in Residually Finite Groups
preprint available at arXiv 1711.04866 [math.GR]

A. Turull
Fitting Height of Groups and of Fixed Points
Journal of Algebra, **86** (1984), 555–566

D. J. S. Robinson
A Course in the Theory of Groups
2nd Edition, Springer–Verlag, 1995

D. Gorenstein
Finite Groups
Chelsea Publishing Company, New York, 1980

Thank you for the attention!