Polynomially-bounded Dehn functions of groups

Alexander Olshanskii

Vanderbilt University

and

Moscow State University

Ischia, March 23, 2018

Area of a word vanishing in the group G

Assume that $G = \langle A \mid R \rangle = \langle a_1, \dots, a_k \mid r_1, \dots, r_l \rangle$.

Let $w = w(a_1, \dots, a_k) \in F = F(A)$. Then $w =_G 1$ iff

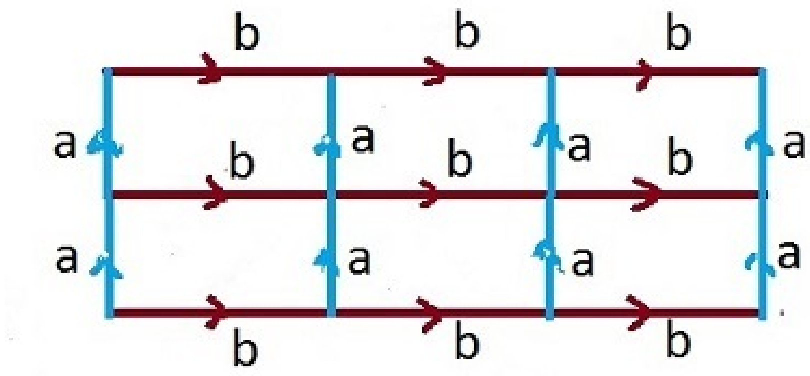
$$w =_F \quad \prod_{i=1}^{t} u_i r_{j_i}^{\pm 1} u_i^{-1}, \ \ where \ \ r_{j_i} \in R \ \ and \ \ u_i \in F$$

$$minimal \ \ number \qquad t = t(w) = Area(w)$$

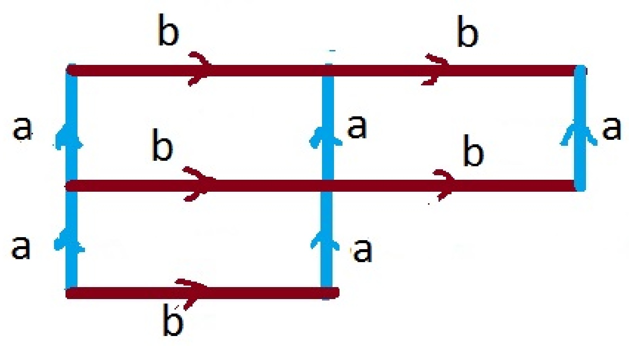
Examples

$$G = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle = \langle a, b \mid ab = ba \rangle$$

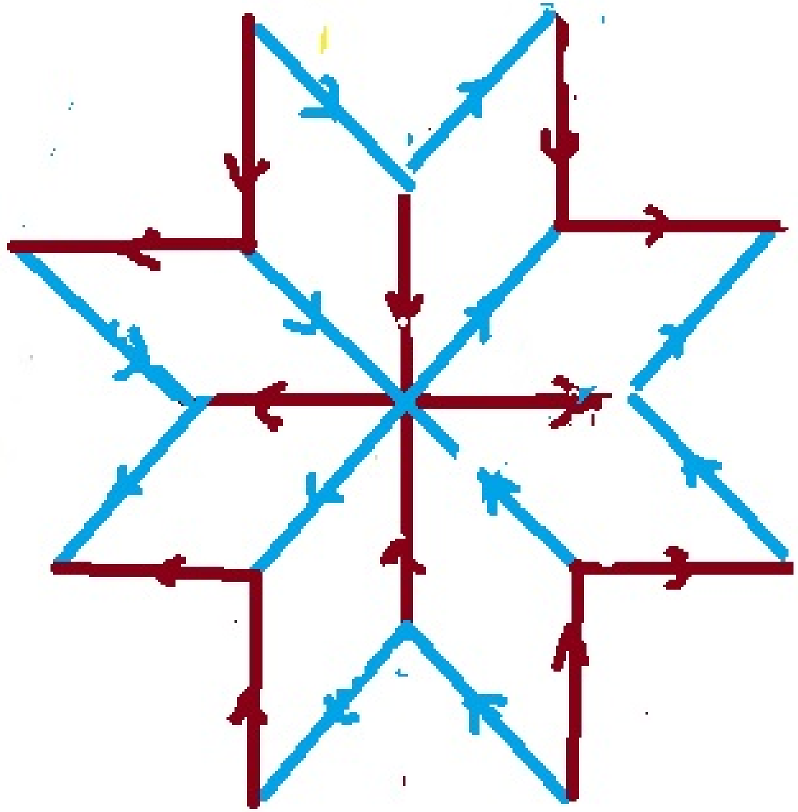
(1) $a^2b^3a^{-2}b^{-3} =_G 1$ and $Area(a^2b^3a^{-2}b^{-3}) = 6$



(2) $a^2b^2a^{-1}b^{-1}a^{-1}b^{-1} =_G 1$ and $Area(a^2b^2a^{-1}b^{-1}a^{-1}b^{-1}) = 3$



(3) $(a^2b^2a^{-2}b^{-2})^2 =_G 1$ and $Area((a^2b^2a^{-2}b^{-2})^2) = 8$



Definition. A van Kampen **diagram** Δ over a presentation $G = \langle A \mid R \rangle$ is a finite, labeled, planar, connected and simply connected 2-complex such that

- For every edge e, $Lab(e) \in A^{\pm 1}$ and $Lab(e^{-1}) = Lab(e)^{-1}$;
- The boundary label of every face Π is a word from $R^{\pm 1}$

Lemma (van Kampen). A word w in the alphabet $A^{\pm 1}$ is equal to 1 in $G = \langle A \mid R \rangle$ iff there exists a diagram Δ over G with boundary label w.

A diagram is called **minimal** if for a fixed boundary label w, it has minimal number of faces. This number is equal to $Area(w)$.

Dehn function of a finitely generated group G

$$d(n) = \max(Area(w) \mid w =_G 1 \;\; and \;\; |w| \leq n)$$

Example: $\langle a \mid \quad \rangle \cong \langle a, b \mid ab \rangle$

$d_1(n) = 0$ but $d_2(n) = \lfloor \frac{n}{2} \rfloor$ since $Area(ab)^m = m$

Equivalence. Given two functions $f, g : \mathbb{N} \to \mathbb{N}$, we define $f \preceq g$ if for some positive integer C and every n, we have $f(n) \leq Cg(Cn) + Cn$. We say that $f \sim g$ if both $f \preceq g$ and $g \preceq f$ hold.

Up to this equivalence, $d(n)$ does not depend on a finite presentation $\langle A \mid R \rangle$ of G. (A group $G = \langle A \mid R \rangle$ is called finitely presented if both A and R are finite sets.)

To prove this one uses Titze transformations of group presentations.

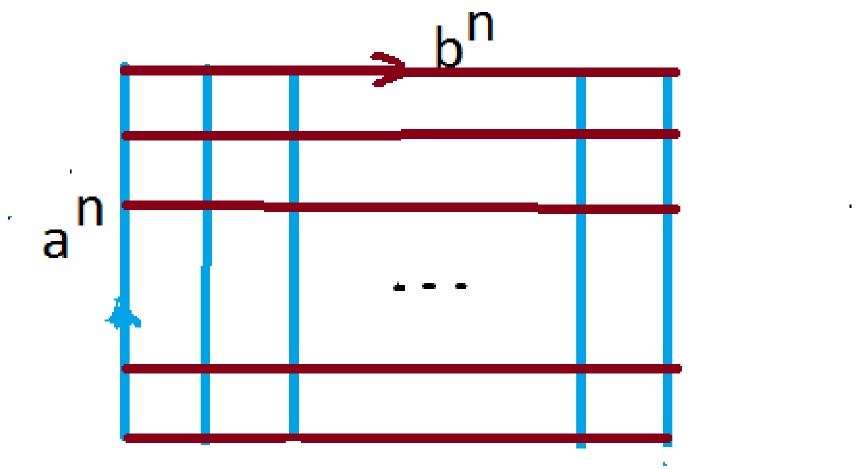
Exercise:
The Dehn function of $\mathbb{Z}^2 \cong \langle a, b \mid aba^{-1}b^{-1} \rangle$ is quadratic

(1) A tip for a quadratic upper bound:

Prove that for every word $w = w(a, b)$ of length at most n there is a derivation $w \to \cdots \to a^k b^l$
with $\leq n^2$ elementary transformations, and $k = l = 0$ if $w =_G 1$

(2) A tip for a quadratic lower bound:

Consider the words $w_n = a^n b^n a^{-n} b^{-n}$ $(n = 1, 2, \dots)$ and prove that $Area(w_n) = n^2$, i.e., prove that the following diagrams are minimal:



Proposition The following properties of a finitely presented group G are equivalent

(a) the Dehn function of G is recursive;

(b) the Dehn function is bounded from above by a recursive function;

(c) the algorithmic word problem is decidable for G.

(c)$\Rightarrow$ (b)
(1) For every word w of length $\leq n$, one can decide whether it trivial or nontrivial in G.

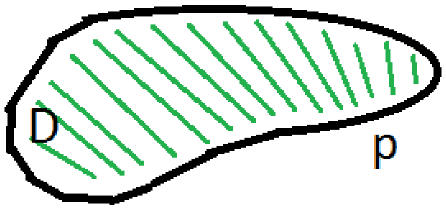
(2) For every trivial word, one can find a presentation $w =_F \prod_{i=1}^{t} u_i r_{j_i}^{\pm 1} u_i^{-1}$ and bound $area(w)$ from above.

(3) This gives a recursive upper bound for the Dehn function.

Recall that there exist finitely presented groups with undecidable word problem (P.S.Novikov, W.W.Boone)

Isoperimetric function of a simply connected Riemannian manifold M

For a smooth simple curve p in M, there is a 'pellicle' (or 'disk') bounded by p such that $Area(D) \leq f(length\ \ of\ \ p)$



Proposition Let G be a finitely generated group isometrically acting on a Riemannian manifold M. If the action is proper and co-compact, then $d_G \sim f_M$.

Examples. (1) $\mathbb{Z}^2$ acts on $\mathbb{R}^2$, $\quad f_{\mathbb{R}^2}(x) = \frac{x^2}{4\pi},\ \ and\ \ f_{\mathbb{Z}^2}(n) \sim n^2$

(2) $G = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$ acts on the standard hyperbolic plane. Therefore G has linear Dehn function.

Groups with linear Dehn function are called (Gromov) hyperbolic.

More examples.

(1) Every finitely generated nilpotent group has at most polynomial Dehn function.

---

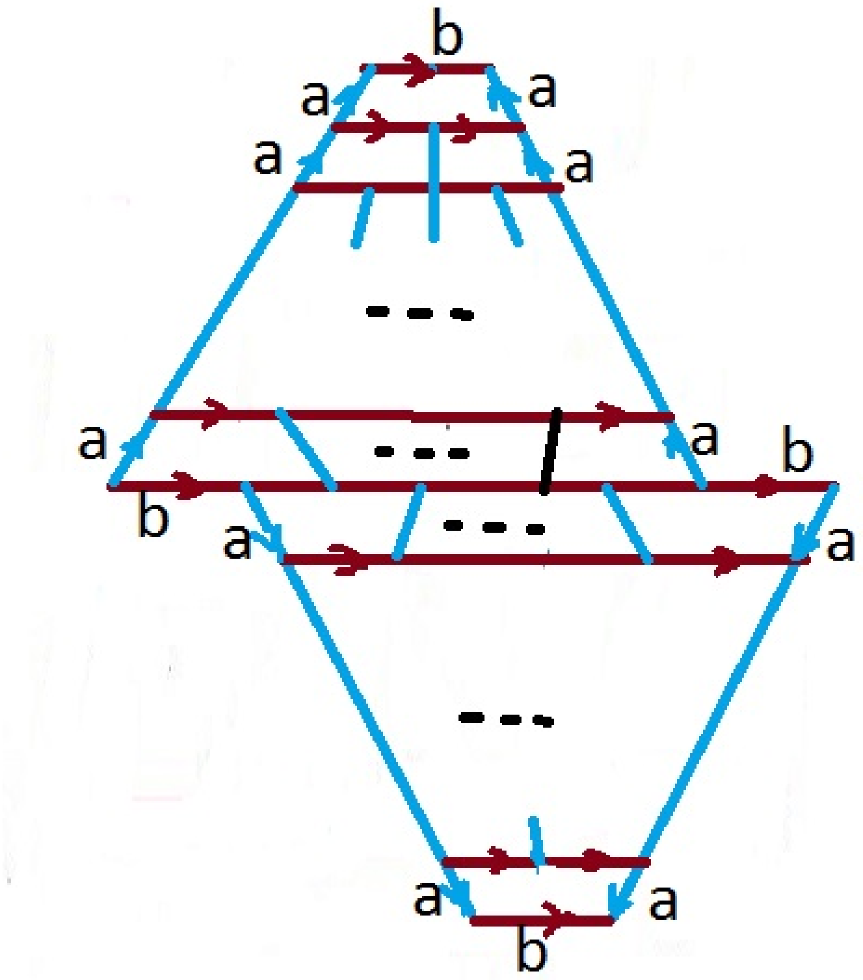
(2) The Dehn function of the one-relator group
$\langle a, b \mid (aba^{-1})b(aba^{-1})^{-1} = b^2 \rangle$
asymptotically exceeds any multi-exponential function
(but still recursive).

---

(3) $G = \langle a, b \mid aba^{-1} = b^2 \rangle = \langle a, b \mid aba^{-1}b^{-2} \rangle$
G has a faithful matrix representation: $a \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $b \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

The minimal diagram for the equality

$$a^n b a^{-n} b a^n b^{-1} a^n b^{-1} =_G 1$$



$$Area(a^n b a^{-n} b a^n b^{-1} a^n b^{-1}) = 2(1 + 2 + \cdots + 2^{n-1}) = 2^{n+1} - 2$$

How large is the set of Dehn functions ?

Question: Are there Dehn functions of finitely presented groups equivalent to n^α, where α is not integer (e.g., $\alpha = \frac{5}{2}$) ?

N. Brady, M.Bridson, (2000): For any pair of positive integers $p > q$, there is a finitely presented group with Dehn function $\sim n^\alpha$, where $\alpha = 2\log_2(\frac{2p}{q})$.

Let M be a Turing machine (deterministic or non-deterministic) accepting a language L. Then for every word $w \in L$, we have $Time(w)$ that is the length of the computation accepting w

Let L_n be the set of all accepted words of length $\leq n$.

Time function (or time complexity)

$$T(n) = \max_{w \in L_n} Time(w)$$

Theorem (M.Sapir, J.-C. Birget, E.Rips, 2002) Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

(a) $f(m+n) \geq f(m) + f(n)$ for any $m, n \in \mathbb{N}$ and

(b) the function $\sqrt[4]{f(n)}$ is equivalent to a time function of a Turing machine (in particular, $f(n) \geq n^4$).

Then there is a finitely presented group with Dehn function equivalent to $f(n)$.

Examples of Dehn functions of groups.

n^α for any algebraic real number $\alpha \geq 4$

$n^{\pi+\sqrt{e}}$

$n^k(\log n)^l$,

$n^k(\log n)^l(\log\log n)^m$, for natural exponents k, l, m ($k \geq 4$)

$\ldots$

Exponents computable in reasonable time

A real number α is computable with time $\leq f(m)$ if there exists a Turing machine which, given a natural number m, computes a binary rational approximation of α with an error $O(2^{-m})$, and the time of this computation $\leq f(m)$.

Corollary (Sapir, Birget, Rips) For a real number $\alpha \geq 4$, the function n^α is equivalent to the Dehn function of a finitely presented group if α is computable with time $\leq 2^{2^m}$.

If $d(n) = o(n^2)$, then $d(n) = O(n)$, that is G is hyperbolic (Gromov, A.O., Bowditch)

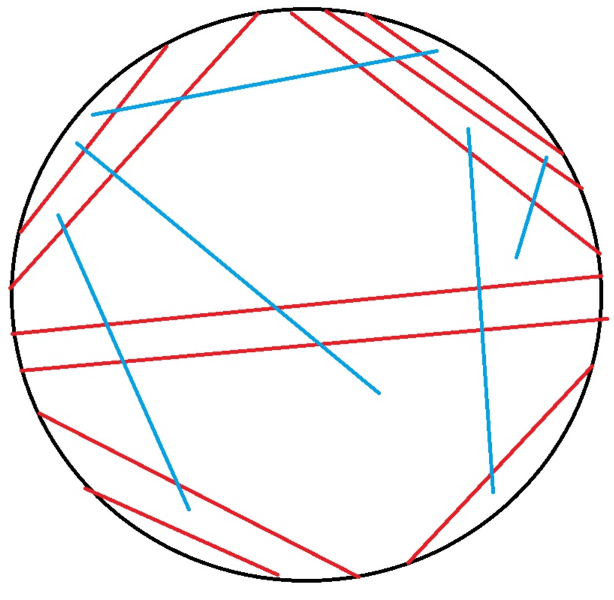
What about the function n^α for $2 < \alpha < 4$?

Theorem (A.O., submitted) If $\alpha \geq 2$ and α is computable with time $\leq 2^{2^m}$, then there is a finitely presented group with Dehn function equivalent to n^α.

The functions $n^\alpha(\log n)^\beta$, $n^\alpha(\log n)^\beta(\log\log n)^\gamma$, etc., are also Dehn functions of finitely presented groups if the exponents α, β, γ are computable in reasonable time.

Corollary, A.O. If if a real number $\alpha \geq 2$ is computable with time $\leq 2^{2^m}$, then there exists a closed connected Riemannian manifold X such that the isoperimetric function of the universal cover $\tilde{X}$ is equivalent to n^α.

By $\mathcal{D}$, we denote Euclidean closed disk of radius 1. Let $\mathbf{T}$ be a finite set of disjoint chords and $\mathbf{Q}$ a finite set of disjoint segments inside $\mathcal{D}$. A segment $Q \in \mathbf{Q}$ and a chord $T \in \mathbf{T}$ may share at most one point.



We say that the pair $(\mathbf{T}, \mathbf{Q})$ is a design.

The length $\ell(Q)$ of Q is the number of the chords crossing Q.

By definition, a segment Q_1 is parallel to a segment Q_2, and we write $Q_1 \parallel Q_2$ if every chord crossing Q_1 also crosses Q_2.

Definition. Given $\lambda \in (0;1)$ and an integer $n \geq 2$, the property $P(\lambda, n)$ of a design says that for any n different segments $Q_1, \ldots, Q_n$, there exist no subsegments $P_1, \ldots, P_n$, respectively, such that $\ell(P_i) > (1-\lambda)\ell(Q_i)$ for every $i = 1, \ldots, n$ and $P_1 \parallel P_2 \parallel \cdots \parallel P_n$.

Lemma (A.O.) There is a constant $C = C(\lambda, n)$ such that for any design $(\mathbf{T}, \mathbf{Q})$ with property $P(\lambda, n)$, we have

$$\sum_{Q \in \mathbf{Q}} \ell(Q) \leq C(\#\mathbf{T}),$$

where $\#\mathbf{T}$ is the number of chords in $\mathbf{T}$.