Identifying groups with a large p-subgroup

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March 22, 2018

Theorem 1 (The Classification of the Finite Simple Groups)

A **finite simple group** *is isomorphic to one of the following groups:*

- *a cyclic group of prime order.*
- *an alternating group* $\mathrm{Alt}(n)$ *of degree* n *at least* 5*.*
- *a finite simple group of Lie type defined in characteristic* p *for some prime* p*.*
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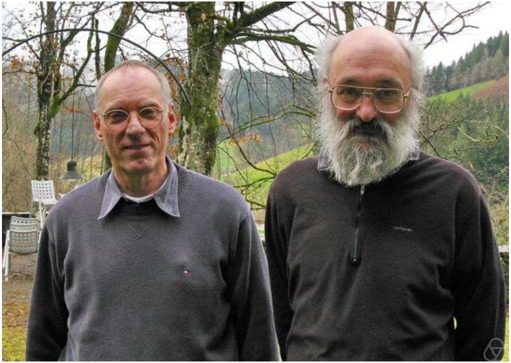
So perhaps we should just attempt to classify them.
The project which I'll describe includes contributions by Mainardis, Meierfrankenfeld, Parmeggiani, Stellmacher, Stroth and others.
Particularly, all of the work here is joint with Gernot Stroth.



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A group in which every p-local subgroup is a $\mathcal{K}$-group is called a $\mathcal{K}_p$-**group.**

If $T \leq G$ is a non-trivial p-subgroup of G, then $N_G(T)$ is called a p-**local subgroup** of G.

All the p-local subgroups H in simple groups of Lie type in characteristic p satisfy

$$C_H(O_p(H)) \leq O_p(H).$$

(This is the Borel-Tits Theorem.)

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Definition 3

A p-local subgroup $H \leq G$ has **characteristic** p if and only if

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Example $X = \mathrm{Sym}(8) \cong \mathrm{SL}_4(2).2$ has parabolic characteristic 2 but it does not have local characteristic 2. In fact

$$C_X((1,2)) \cong 2 \times \mathrm{Sym}(6)$$

and this group does not have characteristic 2 (but it does not contain a Sylow 2-subgroup of X).

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Suppose that Q is a large p-subgroup of G and $Q \leq S \in \mathrm{Syl}_p(G)$. Then Q is normal in S and G has parabolic characteristic p.

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Apart from $\mathrm{Sp}_{2n}(2^a)$, $\mathrm{F}_4(2^a)$ and $\mathrm{G}_2(3^a)$, all simple Lie type groups defined in characteristic p have a large p-subgroup.

Example 6

Let $G = \mathrm{McL}$. Then G has local characteristic 3, all 3-local subgroups are conjugate into subgroups of shape

$$X = 3^4 : \mathrm{Mat}(10) \text{ or } Y = 3_+^{1+4} : 2.\mathrm{Sym}(5)$$

and $O_3(Y)$ is a large 3-subgroup of G. This group contains a subgroup $H \cong \mathrm{PSU}_4(3)$ containing the Sylow 3-subgroup and so it's almost of Lie type.

Developments since the classification theorem was announced to be very close to complete on June 22, 1980 ("Mathematics: A School of Theorists Works Itself Out of a Job") include:

- The **amalgam method** for controlling the structure of p-local subgroups (post 1980) uses the coset graph to keep control weak closure arguments.

Developments since the classification theorem was announced to be very close to complete on June 22, 1980 ("Mathematics: A School of Theorists Works Itself Out of a Job") include:

- The **amalgam method** for controlling the structure of p-local subgroups (post 1980) uses the coset graph to keep control weak closure arguments.
- Determination of small modules with nice actions of p-subgroups for quasisimple groups.

Let G be quasisimple, V a faithful $\mathrm{GF}(p)G$-module and $A \leq G$ is an elementary abelian p-subgroup.

- More results about **quadratic pairs**. Chermak has the most useful results (1999,2002, 2004). For odd primes p, he determines the triples (G, V, A) with

$$[V, A, A] = 0$$

  for groups G which are not Lie type in characteristic p odd primes p.

- Determination of F-**modules** and $2F$-**modules** (McLaughlin (1967), Cooperstein (1978), Meixner (1991), Guralnick-Malle (2002-2003), Mierfrankenfeld-Stellmacher (2009)). Triples (G, V, A) with

$$|V : C_V(A)| \leq |A| \text{ (F-module)}$$

  and

$$|V : C_V(A)| \leq |A|^2 \text{ (2F-module).}$$

The main steps in the programme are:

Step 1 Understand the structure of the p-local subgroups which contain a fixed Sylow p-subgroup of G.

Step 2 Create an almost simple subgroup H of G containing S such that H has known isomorphism type.

Step 3 Prove that $G = H$.

The main contribution towards Step One is the so-called Structure Theorem.

Theorem 7 (Meierfrankenfeld-Stellmacher-Stroth: The Local Structure Theorem (AMS Memoir 2016))

Suppose that

- G *is a* $\mathcal{K}_p$*-group which is almost simple group;*
- $S \in \mathrm{Syl}_p(G)$ *and* $Q \leq S$ *is large;*
- S *is contained in at least* 2 *maximal p-local subgroups of* G*;*
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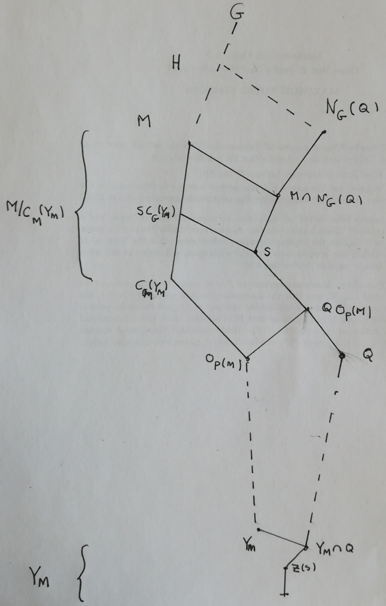
- *a further p-local condition.*

Let M *be a maximal p-local subgroup of* G *not contained in* $N_G(Q)$ *and* $Y_M = \Omega_1(Z(O_p(M)))$*. Then the possibilities for the pair*

$$(M/C_M(Y_M), Y_M)$$

are known.

The Local Structure Theorem



The further investigation divides into two parts. Set

$$\mathcal{L}_G(S) = \{X \leq G \mid S \leq X \text{ and } O_p(X) \neq 1\}.$$

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- (a) There exists a p-local subgroup $M \in \mathcal{L}_G(S)$ such that $\langle Y_M^{N_G(Q)} \rangle$ is non-abelian.
- (b) For all p-local subgroup $M \in \mathcal{L}_G(S)$ we have $\langle Y_M^{N_G(Q)} \rangle$ is abelian.

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Most of the groups arise is cases (1) and (2)(a) and typically (2)(b) leads to a contradiction.

The result of case (1) is a theorem called the H-structure.

Theorem 8 (The H-Structure Theorem, Meierfrankenfeld-P-Stroth, 2018+)

Suppose there exists a p-local subgroup $M \in \mathcal{L}_G(S)$ with $M \not\leq N_G(Q)$ and $Y_M \not\leq Q$. Then there exists $M^* \in \mathcal{L}_G(S)$ and L with $S \leq L \leq N_G(Q)$ such that setting

$$H = \langle M^*, L \rangle$$

either $F^*(H)$ is a simple group of Lie type in characteristic p or $F^*(G)$ is a known finite simple group.

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This theorem establishes Step 2 when Case (1) holds.

1. The Local Structure Theorem is used to show that the assumption $Y_M \not\leq Q$ implies that there exists $M^* \in \mathcal{L}_G(S)$ such that

$$M^*/C_{M^*}(Y_{M^*}) \cong \Omega_n^{\pm}(q),$$

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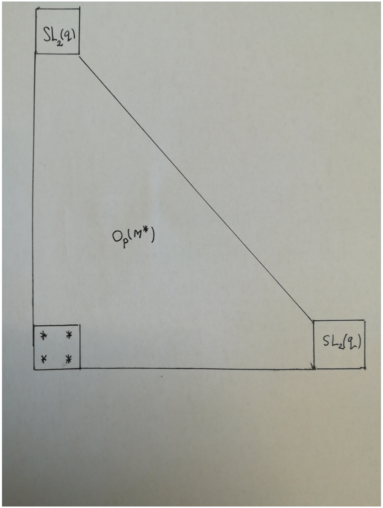
5. Deduce that either $F^*(H)$ is a Lie type group in characteristic p, or H a weak BN-pair or we know G.

Assume that $M^*/C_{M^*}(Y_{M^*}) \cong \Omega_4^+(p^a)$ and Y_M has order p^{4a} is the natural $M^*/C_{M^*}(Y_{M^*})$-module.



1. We have

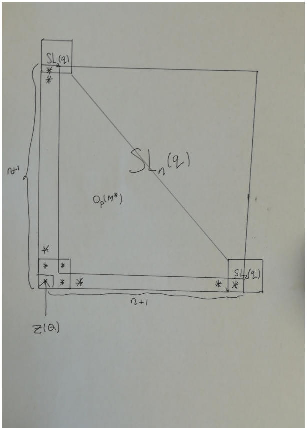
$$|Y_{M^*}Q/Q| = p^a,$$

$$|[Y_{M^*}, Q]| = p^{3a}, \text{ and } |C_{Y_{M^*}}(Q)| = p^a.$$

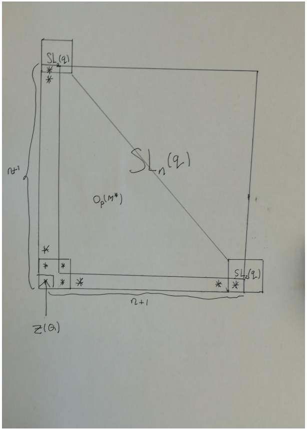
2. So $Q/Z(Q)$ is a $2F$-module. Use this to eventually show that

$$L^*/Q \cong \mathrm{SL}_n(p^a)$$

for some $n \geq 2$ and $Q/Z(Q)$ is a direct sum of two natural $\mathrm{SL}_n(p^a)$-modules



SL2(q)
SLn(q)
Op(M*)
n+1
n+1
SL2(q)
Z(G)



Recover all the parabolic subgroups of $\mathrm{PSL}_{n+2}(p^a)$ containing S and deduce that

$$H \cong \mathrm{PSL}_{n+2}(p^a)$$

by using a version of Tit's Local Approach Theorem by Meierfrankenfeld, Stroth and Weiss (Proc. Cam. Phil. Soc. 2013).

But

But perhaps

$$H \neq G.$$

☹

Assume that $F^*(H)$ **is a group of Lie type in characteristic** p **of rank at least** 2 **and** $O_p(G) = 1$**.**

The group H could be **strongly** p**-embedded** in G. This is equivalent to saying $N_G(T) \leq H$ for all non-trivial p-subgroups of G.

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Theorem 9 (Bender $p = 2$, P & Stroth 2011 $p \neq 2$)

Suppose that G *is a* $\mathcal{K}_{p,2}$*-group. Then* H *is not strongly* p*-embedded unless perhaps* $F^*(H) \cong \mathrm{PSL}_3(p)$ *with* p *odd.*

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Problem: Explain why in a $\mathcal{K}_{p,2}$-group, $\mathrm{PSL}_3(p)$ cannot be strongly p-embedded.

Theorem 10 (P, Pientka,Seidel & Stroth (Submitted 2017))

Suppose that p is a prime, G is a $\mathcal{K}_{p,2}$*-group of local characteristic p and H is a subgroup of G of index coprime to p. Assume that* $H = N_G(F^*(H))$ *and* $F^*(H)$ *is a simple group of Lie type in characteristic p and of rank at least two. Then either* $G = H$ *or one of the following holds*

1. $p = 2$ *and* $F^*(G) \cong \mathrm{Mat}(11)$, $\mathrm{Mat}(23)$ *or* $\mathrm{G}_2(3)$*;*
2. $p = 3$ *and* $F^*(G) \cong \mathrm{McL}$*;*
3. $p = 5$ *and* $G \cong \mathrm{Ly}$*; or*
4. p *is odd and* $F^*(H) \cong \mathrm{PSL}_3(p)$*.*