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Composition lengths of finite groups

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Measures of size of a group

- c(G) = composition length of G [G finite group]
  = number of composition factors, counting multiplicities
- For example, c(G) = 3 for G = $C_2 \times C_2 \times C_2$

- Other measures: |G| = order of group
- d(G) = minimum size of generating set for G

Inter-related measures

- $d(G) \le 2\, c(G)$ and $c(G) \le \log_2 (|G|)$

Most useful if estimate in terms of parameters relevant to the way G is represented. E. g.

- function of n if $G \le \mathrm{Sym}(n)$ or
- function of n and q if $G \le GL(n,q)$

Many examples of such bounds in literature

The groups we – and others - studied: finite groups

- Permutation groups G ≤ Sym(n) – G arbitrary, transitive, primitive

- Linear groups G ≤ GL(n, q) – G irreducible or completely reducible on V

G ≤ Sym(n) primitive if the only G-invariant partitions are trivial

G ≤ GL(n, q) completely reducible if V = $V_1 \oplus V_2 \oplus \cdots \oplus V_r$ such that each V_i is G-invariant and irreducible

Composition length bounds in the literature

For G ≤ Sym(n), with s orbits

1974 Fisher

$c(G) \leq \frac{4(n-s)}{3}$ and shows the bound achieved by transitive action of degree $n = 4^k$ of $G = Sym(4) \wr Sym(4) \wr \cdots \wr Sym(4)$

1993 Pyber (states) for G primitive and not Sym(n) or Alt(n)

$c(G) \leq c \log_2 n$ (and $b(G) \leq \log_2 n$)

For G ≤ GL(n,q) completely reducible, with $q = p^f$

2001 Lucchini, Menegazzo and Morigi

$c(G) \leq c\, n \log_2 q$

Proof of Pyber's bound appeared only in 2017 ! Guralnick, Maroti and Pyber

With c = 2+ $\log_9(48 \text{ x } 24^{1/3})$
≈ 4.244

Composition length questions

Stephen Glasby's questions

1974 Fisher linear bound for c(G) is sharp for (transitive) permutation groups

1993 Pyber bounds for c(G) for primitive G < Sym(n) – best constant? sharp?

2001 Lucchini, Menegazzo and Morigi – best constant for c. r. linear groups?

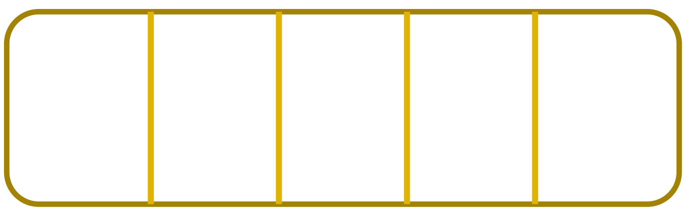
Could we find sharp upper bounds for c(G) for primitive subgroups G of Sym(n)

And could we classify all groups attaining bounds?

Composition length questions

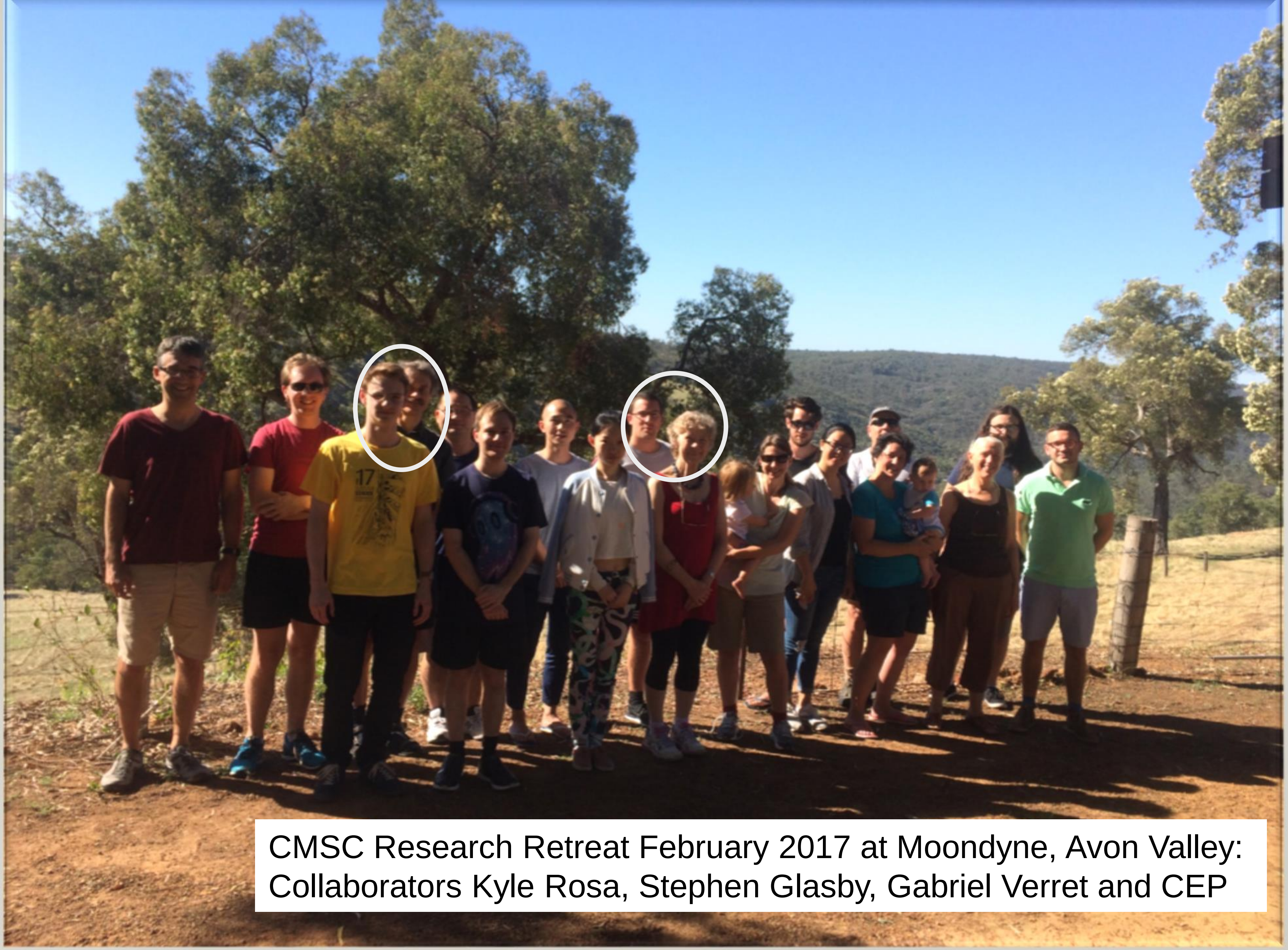
2017 Stephen Glasby's questions: permutation groups

Could we find sharp upper bounds for c(G) for primitive subgroups G of Sym(n)



G may leave a partition invariant

G imprimitive on X means G preserves a nontrivial partition P of X
G primitive: only G-invariant partitions have |P|=1 or parts of size 1



CMSC Research Retreat February 2017 at Moondyne, Avon Valley:
Collaborators Kyle Rosa, Stephen Glasby, Gabriel Verret and CEP

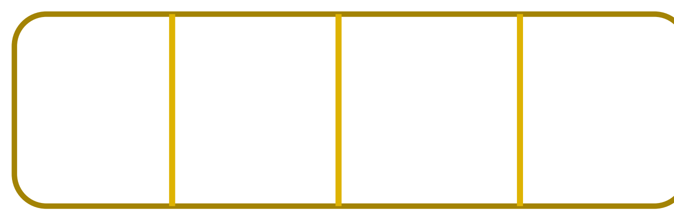
Composition length questions

1974 Fisher's Groups

Consider the extreme examples of transitive groups from Fisher's1974 paper.

- First group $G = Sym(4)$ on $N(1)$ with $n = 4$ and $c(G) = 4 = \frac{4(4-1)}{3}$
- Second group $G = Sym(4) \wr Sym(4) = Sym(4)^4.Sym(4)$ acts on $N(2)$ with $n = 16$ and $c(G) = 20 = \frac{4(16-1)}{3}$
- And so on. Call the k^{th} group $G = T_k = Sym(4) \wr T_{k-1}$ acts on $N(k) = N(1) \times N(k-1)$ so $n = 4.\left(4^{k-1}\right) = 4^k$ and $c(G) = \frac{4\left(4^k-1\right)}{3} = \frac{4(n-1)}{3}$

Think of $T_k = Sym(4) \wr T_{k-1}$ preserving partition of $N(k)$ with blocks of size 4 and block set labelled by the set that T_{k-1} acts on.



Composition length questions

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2017 Stephen's Insights

Consider the same groups with a different (primitive) action (product action)

- First group $P_1 = Sym(4)$ on $X(1) = N(1)$ with $n = 4$ and $c(P_1) = 4$
- Second group $P_2 = Sym(4) \wr Sym(4)$ acts on $X(2) = N(1)^{N(1)}$ of size $n = 4^4$ and $c(P_2) = 20$
- Third group $P_3 = Sym(4) \wr T_2$ acts on $X(3) = N(1)^{N(2)}$ of size $n = 4^{4^2}$
- The $(k+1)^{st}$ group $P_{k+1} = Sym(4) \wr T_k$ acts on $X(k+1) = N(1)^{N(k)}$ of size $n = 4^{4^k}$ and $c(P_{k+1}) = \frac{4(4^{k+1}-1)}{3} = \frac{8(log_2 n - 2)}{3}$

What is going on?

- All these actions primitive
- Logarithmic relation between n and c(G) for G = P_{k+1} [Pyber: $c(G) \leq c \log_2 n$]
- Why would we guess that these might be extreme examples?

Composition length results

Theorem 1:

- For general permutation groups $G \leq Sym(n)$ with s orbits

- Fisher

$$c(G) \leq \frac{4}{3}(n-s)$$

- GPRV

  Equality holds if and only if $G = G_1 \times \cdots \times G_s$
  group G_i induced on the i^{th} orbit is $G_i = T_{k_i}$ of degree 4^{k_i}

And what about the proof?

- Induction on the degree n quickly reduces to the case where G is primitive.
- Use MAGMA to check n ≤ 24. For n > 24, we use Maroti's result –
- If primitive $G \neq Sym(n), Alt(n)$ then $|G| \leq 2^{n-1}$, so $c(G) \leq n-1 < \frac{4}{3}(n-1)$.
- If $G = Sym(n), Alt(n)$ then $c(G) < \frac{4}{3}(n-1)$ unless $n = 4$ and $G = Sym(4)$

Composition length results

Theorem 2 (GPRV):

- For primitive permutation groups $G \leq Sym(n)$

$$c(G) \leq \frac{8}{3}\log_2 n - \frac{4}{3}$$

And equality holds if and only if $n = 4^{4^k}$ for some $k \geq 0$, and $G = P_k$ in product action

So yes indeed, optimal Pyber c is $\frac{8}{3}$!

And the proof?

- Induction on the degree n using the O'Nan—Scott Theorem
- The affine case required a result about linear groups – for the bound and to identify the extreme examples

Composition length results

Theorem 3 (GPRV) :

- For completely reducible linear groups $G \leq GL(n,q)$ such that G has s irreducible constituents in $V = V_1 \oplus \cdots \oplus V_s$

Let $q = p^f$ with p prime and $f \geq 1$. Then

$$c(G) \leq \left(\frac{8}{3}\log_2 p - 1\right) nf \; - s\left(\log_2 f + \frac{4}{3}\right)$$

If $q = 2$ this is $c(G) \leq \dfrac{5n - 4s}{3}$

And equality holds if and only if $p = 2$, $G = G_1 \times \cdots \times G_s$, where G_i is the group induced on V_i and either

- $q = 2$ and $G_i = GL(2,2) \wr T_{k_i} \leq GL(n_i, 2)$ with $n_i = \dim V_i = 2^{2k_i+1}$, or
- $q = 4$ and $G_i = GL(1,4) \cong C_3$ with $n_i = \dim V_i = 1$

So optimal constant c for Lucchini, Menegazzo, Morigi 2001 is also c = 8/3

Composition length more questions

General permutation groups: $c(G) < \frac{4}{3}n$ where *n* is the degree

Primitive permutation groups: $c(G) < \frac{8}{3}\log_2 n$

For what classes of permutation groups do we get a logarithmic bound $c(G) \leq c \ log_2 n$ for some constant *c*?

Quasiprimitive and semiprimitive permutation groups

Quasiprimitive: each nontrivial normal subgroup transitive

Naturally arise when studying arc transitive graphs

Semiprimitive: each normal subgroup transitive or semiregular

Natural example: GL(n,q) on nonzero vectors

Composition length: semiprimitive groups

Primitive $\Rightarrow$ quasiprimitive $\Rightarrow$ semiprimitive

Semiprimitive but not quasiprimitive $G \leq Sym(n)$ of degree n

Theorem (GPRV) : $c(G) \leq \frac{8}{3}\log_2 n - 3$

Infinitely many extreme semiprimitive examples exist – examples very similar to the extreme primitive examples all "sort of affine type"

Turns out something very different happens for quasiprimitive but not primitive groups

Composition length: quasiprimitive groups

Quasiprimitive, but not primitive, group $G \leq Sym(n)$ on X

Choose G-invariant partition Y of X with part size maximal.
Number of parts $m \mid n$ and $G \cong$ primitive permutation group on Y.

This implies $c(G) < \frac{8}{3}\log_2 m \; < \frac{8}{3}\log_2 n$

Extreme primitive examples all "affine type" while (provably) quasiprimitive but not primitive groups are NOT of "affine type"

What is the best constant c such that $c(G) < c \; log_2 n$ for primitive groups not of "affine type" ?

Composition length: quasiprimitive groups

Quasiprimitive, but not primitive, group $G \leq Sym(n)$ on X

Choose G-invariant partition Y of X with part size maximal.
Number of parts $m \mid n$ and $G \cong$ primitive permutation group on Y.

This implies $c(G) < \frac{8}{3}\log_2 m < \frac{8}{3}\log_2 n$

$c = \frac{8}{3} \approx 2.66667$

Theorem (GPRV):

For primitive permutation groups $G \leq Sym(n)$ not of affine type

$$c(G) \leq c' \log_2 n - \frac{4}{3} \quad \text{where} \quad c' = \frac{10}{3\,log_2 5} \approx 1.43559$$

And equality holds if and only if $n = 5^{4^k}$ and $G = Sym(5) \wr T_k$ in product action for some $k \geq 0$

Composition length: quasiprimitive groups

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Quasiprimitive, but not primitive, group $G \leq Sym(n)$ on X

Choose G-invariant partition Y of X with part size maximal.
Number of parts $m \mid n$ and $G \cong$ primitive permutation group on Y.

Immediate Corollary (GPRV):
For quasiprimitive but not primitive groups $G \leq Sym(n)$

$$c(G) < c' \log_2 n \quad \text{where} \quad c' = \frac{10}{3 \log_2 5} \approx 1.43559$$

But equality (probably) never holds.
We have examples which imply that the optimal constant c' satisfies $0.73585 \approx \frac{31}{12 \log_2 5 + 9 \log_2 3} \leq c' \leq 1.43559$

Results summary

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Theorems:

- For $G \leq Sym(n)$ or $G \leq GL(n, p^f)$ (completely reducible) we find sharp upper bounds for $c(G)$ in terms of n or n, p, f
- For $G \leq Sym(n)$ primitive, we determine the optimal constant c such that $\mathrm{c}(G) \leq c\ log_2 n$ namely $c = \frac{8}{3}$
- We classify all examples attaining these upper bounds

Logarithmic bounds:

For classes of semiprimitive and quasiprimitive permutation groups we also get a logarithmic bound $\mathrm{c}(G) \leq c\ log_2 n$

Open questions as to

- The best constant c for quasiprimitive groups
- Just what classes are at the borderline between log and linear bounds