# Metanilpotent Groups Satisfying the Minimal Condition on Normal Subgroups

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The following results are well known.

**Theorem.** If $H$ is a subgroup with finite index in a group $G$ with min-$n$, then $H$ has min-$n$. (J.S. Wilson).

**Theorem.** *A soluble group with min-n is locally finite.* (R. Baer).

Metanilpotent groups with min-$n$ were first studied by D. McDougall.

The next result is basic.

**Theorem.** *A metanilpotent group with min-n is countable.* (H.L. Silcock).

The proof reduces quickly to the metabelian case, which is due to McDougall.

On the other hand, B. Hartley has constructed uncountable soluble groups of derived length 3 with min-$n$.

# Čarin modules

The first example of a metabelian group with min-$n$ that is not a Černikov group was given by V.S. Čarin.

Let $p$ be a prime and $\pi$ a finite set of primes with $p \notin \pi$. The algebraic closure $K$ of $\mathbb{Z}_p$ contains primitive $q^i$th roots of unity for $q \in \pi$, $i = 1, 2, \dots$.

Let $Q$ be the subgroup of $K^*$ generated by all these roots and let $F$ be the subfield of $K$ generated by $Q$. Then $F$ is a $Q$-module via the field operations.

# Čarin groups

It is easy to prove that $F$ is a simple $Q$-module. Call $F$ the *Čarin* $(p, \pi)$*-module* over $Q$. The semidirect product

$$G = Q \ltimes F$$

is the *Čarin group of type* $(p, \pi)$. This is a metabelian group with min-$n$, which is not Černikov. Then $Q \simeq \mathrm{Dr}_{q \in \pi}\, q^{\infty}$ and $F$ is an elementary abelian $p$-group.

Let $G$ be a metanilpotent group with min-$n$ and let $N \lhd G$ where $N$ and $Q = G/N$ are nilpotent. Then $A = N^{ab}$ is an artinian module over the nilpotent Černikov group $Q$.

# Artinian modules over nilpotent Černikov groups

Let $A$ be an artinian module over the nilpotent Černikov group $Q$. Here are two useful results.

**Lemma.**

(i) *$A$ is countable and periodic as an abelian group.*

(ii) *There is an expression $A = D + B$ where $D$ and $B$ are $Q$-submodules, $D$ is divisible and $B$ is bounded (that is, of finite exponent) as abelian groups.*

**Proposition.** Let $A_0$ be the largest hypertrivial submodule of $A$. Then $H^n(Q, A/A_0) = 0$ for all $n \geq 0$.

# Module constructions

Hartley and McDougall showed how to construct artinian uniserial modules over locally finite groups from simple modules.

Let $p$ be a prime and $Q$ a countable, locally finite $p'$-group. Let $\{M_\lambda | \lambda \in \Lambda\}$ be a complete set of non-isomorphic simple $\mathbb{Z}_p Q$-modules. Let $M_\lambda$ have rank $r_\lambda$. Choose a divisible abelian $p$-group $V_\lambda$ of rank $r_\lambda$ and identify $M_\lambda$ with $V_\lambda[p]$, so $V_\lambda[p]$ has a $Q$-module structure.

# The modules $V_\lambda(n), V_\lambda(\infty)$

Since $Q$ is a countable locally finite $p'$-group, the module structure of $V_\lambda[p]$ extends to $V_\lambda$. Let the resulting $Q$-module be

$$V_\lambda(\infty).$$

The only proper submodules of $V_\lambda(\infty)$ are $V_\lambda(n) = V_\lambda[p^n]$, where $n = 0, 1, 2, \ldots$. Thus $V_\lambda(\infty)$ is an artinian uniserial $Q$-module. Also

$$V_\lambda(n+1)/V_\lambda(n) \overset{Q}{\simeq} M_\lambda.$$

In addition $V_\lambda(\infty)$ is divisible and is the injective hull of $M_\lambda$.

# The Hartley-McDougall decomposition

**Theorem.** *Let $p$ be a prime and $Q$ a countable locally finite $p'$-group. Let $A$ be an artinian $Q$-module which is a $p$-group. Then $A$ is the direct sum of finitely many artinian uniserial modules (of types $V_\lambda(n), V_\lambda(\infty)$). The direct decomposition is unique up to an automorphism of $A$.*

This can be applied to artinian modules over nilpotent Černikov groups *in the non-modular case*.

In a modular situation the H-M decomposition cannot be used. But here is useful fact.

**Lemma.** *If $A$ is an artinian module over a nilpotent Černikov group $Q$, then $Q_p/C_{Q_p}(A_p)$ is finite for all primes $p$.*

This means that by passing to a suitable subgroup of finite index we reach a non-modular situation.

# Applications

**Proposition.** *Let $A$ be an artinian module over a nilpotent Černikov group $Q$. If $A$ is bounded as an abelian group, then it is $Q$-noetherian.*

*Proof.* Assume $A$ is a $p$-group and $Q$ acts faithfully on it. Then $Q = P \times R$ where $P$ is finite and $R$ is a $p'$-group. Then $A$ is $R$-artinian. By Hartley-McDougall $A$ is a finite direct sum of $R$-modules of the form $V_\lambda(n)$. Each one is $R$-noetherian, so $A$ is $R$-noetherian and hence $Q$-noetherian. □

**Corollary.** *Let $G$ be a metanilpotent group with min-n and let $N \lhd G$ be nilpotent. Then $N'$ has finite exponent and satisfies max-$G$.*

*Proof.* The group $N^{ab}$ is the direct product of a divisible group and a group of finite exponent. By the tensor product property of the lower central series $\gamma_i(N)/\gamma_{i+1}(N)$ has finite exponent for $i \geq 2$, and hence satisfies max-$G$. Since $N$ is nilpotent, $N'$ has finite exponent and max-$G$. □

# Modules in the modular case

We will need information about modules in the modular case: here some level of imprecision is unavoidable.

A module $A$ is the *near direct sum* of submodules $A_i$, $i = 1, 2, \dots, n$ if $A = \sum_{i=1}^{n} A_i$ and $A_i \cap \sum_{j=1, j \neq i}^{n}$ is bounded as an abelian group. Write

$$A = A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n.$$

# A module decomposition in the modular case

**Theorem.** (Arikhan, Cutolo and Robinson). *Let $A$ be an artinian module that is a $p$-group over a nilpotent Černikov group $Q$. Then*

$$A = (A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n) + A[p^\ell]$$

*where the $A_i$ are $p$-adically irreducible $Q$-modules, (i.e., minimal unbounded submodules of type $V_\lambda(\infty)$), and $\ell \geq 0$.*

**Corollary.** $A/B \simeq A_1 \oplus A_2 \oplus + \cdots \oplus A_n$ *where $B$ is a bounded submodule.*

# The Fitting subgroup

**Theorem.** *Let $G$ be a metanilpotent group with min-n. Then the Fitting subgroup of $G$ is nilpotent.*

**Corollary.** *Let $F$ be the Fitting subgroup of a metanilpotent group with min-n . Then there exists $S \triangleleft G$ such that $S$ has max-$G$ and finite exponent, while $F/S$ is the direct sum of finitely many uniserial injective $G/F$-modules of type $V_\lambda(\infty)$.*

# The nilpotent supplement theorem

Let $G$ be metanilpotent with min-$n$ and let $A = \gamma_\infty(G)$, the smallest term of the lower central series. Then $A$ and $G/A$ are nilpotent.

**Theorem.** *There is a nilpotent Černikov subgroup $X$ such that*

$$G = XA$$

*and $X \cap A$ is finite.*

# Groups without proper subgroups of finite index

Let $G$ is a metanilpotent group with min-$n$ and let $G_0$ be its finite residual. Then $G_0 \lhd G$, $G/G_0$ is finite and $G_0$ has no proper subgroups of finite index and satisfies min-$n$.

With $G_0$ we are in the non-modular case.

**Lemma.** *A metanilpotent group $G$ with min-n has no proper subgroups of finite index if and only if $G/\gamma_\infty(G)$ is a divisible abelian group.*

For these groups the nilpotent supplementation theorem takes a sharper form:

**Theorem.** (Silcock). *Let $G$ be a metanilpotent group with min-n which has no proper subgroups of finite index. Then $\gamma_\infty(G) = G'$ and there is a divisible abelian subgroup $D$ of finite rank such that $G = DG'$ and $D \cap G'$ is finite and contained in $Z(G) \cap G''$.*

The special case where $G$ is metabelian is noteworthy.

**Corollary.** (McDougall). *If in addition $G$ is metabelian, then $G = D \ltimes G'$. Also $G'$ is the direct sum of finitely many $G/C_G(A)$-modules of types $V_\lambda(n), V_\lambda(\infty)$ which arise from simple modules.*

This is effectively a classification of these groups.

# Sylow properties

Recall that if $\pi$ is a non-empty set of primes, a *Sylow $\pi$-subgroup* of a group is a maximal $\pi$-subgroup.

A group $G$ is called *Sylow $\pi$-connected* if all the Sylow $\pi$-subgroups are conjugate.

A group $G$ is *Sylow $\pi$-integrated* if every subgroup is Sylow $\pi$-connected.

# Sylow properties

**Theorem.** *A metanilpotent group with min-n is Sylow $\pi$-integrated for all $\pi$.*

This is due to McDougall in the case of metabelian groups.

However, soluble groups with min-*n* and derived length 3 are not Sylow *p*-connected in general (M. Dixon).

# Frattini properties

The Frattini subgroup of a metanilpotent group with min-$n$ need not be nilpotent.

*Examples*

(i) Let $G = \langle x \rangle \ltimes (A \oplus A)$ where $A$ is a $2^\infty$-group, $x^4 = 1$ and $(a_1, a_2)x = (a_2, -a_1)$; here $\phi(G)$ is a 2-group of dihedral type. Note that $G$ is a Černikov group.

(ii) Let $p, q$ be different primes, $Q = q^\infty$, $A$ an injective Čarin $Q$-module of type $(p, q)$. Set $G = Q \ltimes A$. Then $G$ has no maximal subgroups, so $\phi(G) = G$, which is not even locally nilpotent.

**Theorem.** *Let $G$ be a metanilpotent group satisfying min-n and put $A = \gamma_\infty(G)$. Let $D/A'$ denote the maximum divisible subgroup of $A^{ab}$. Then:*

(i) *$\phi(G)$ is nilpotent if and only if $\phi(G/D)$ centralizes $D/A'$.*

(ii) *If $\phi(G)$ is nilpotent, then*

$$\mathrm{Fitt}(G/\phi(G)) = \mathrm{Fitt}(G)/\phi(G)$$

*if and only if $\mathrm{Fitt}(G/D)$ centralizes $D/A'$.*

# Some recent developments

Recently there has been renewed interest in metanilpotent groups with min-$n$ in connection with research on countability restrictions on subgroup lattices. For example, there is the following result of Arikhan, Cutolo and Robinson (2017).

**Theorem.** *A metanilpotent group with min-n has countably many maximal subgroups.*

# Countably dominated groups

A group $G$ is said to be countably dominated (**CD**) if it has a countable set of proper subgroups $\mathcal{S}$ such that every proper subgroup of $G$ is contained in some member of $\mathcal{S}$. If $G$ is a **CD**-group, then it can have only countably many maximal subgroups.

**Theorem.** (A-C-R). *Let $G$ be a metanilpotent group satisfing min-n and write $A = \gamma_\infty(G)$. Then $G$ is countably dominated if and only if $A^{ab}$ has countably many submodules and the finite residual of $G/A$ is locally cyclic.*

The condition on $A^{ab}$ in the theorem can be expressed in terms of the module structure.

**Theorem 8.3.** (A-C-R). *Let $A$ be an artinian module over a nilpotent Černikov group $Q$. Then the following statements are equivalent.*

(i) *$A$ has countably many submodules.*

(ii) $A = A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n + S$ *where the $A_i$ are pairwise non-near isomorphic, p-adically irreducible submodules and $S$ is a bounded submodule.*

Here $A_i$ is *nearly isomorphic with* $A_j$ if there are bounded submodules $B_i, B_j$ such that $A_i/B_i \simeq A_j/B_j$ as $Q$-modules.