On conciseness of words in residually finite groups

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On the other hand, many important words are known to be concise.

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Such words are also known under the name of outer commutator words and are precisely the words that can be written in the form of multilinear Lie monomials, ex.

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Merzlyakov (1967) showed that every word is concise in the class of linear groups while Turner-Smith (1966) proved that every word is concise in the class of residually finite groups all of whose quotients are again residually finite.

The negative solution of Hall's problem was obtained by Ivanov in 1989 by constructing a group G admitting a word w that takes precisely two values in G and has $w(G)$ infinite cyclic. The group G constructed by Ivanov is not residually finite.

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In a sense, the residually finite case of the problem is more interesting than the original version, since it allows use of a greater variety of tools.

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I will now describe some recent results on Hall's problem about verbal subgroups in residually finite groups. In most cases the proofs are based on techniques created by Zelmanov in the solution of the restricted Burnside problem.

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It is unknown whether this word is concise (in the class of all groups).

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Roughly, there are three main ingredients to the solution of the restricted Burnside problem: the classification of finite simple groups, the Hall-Higman theory, and Zelmanov's Lie-theoretic techniques. The above question seems really hard because the Hall-Higman theory does not work here.

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Set $[x, {}_1 y] = [x, y]$ and $[x, {}_{i+1} y] = [[x, {}_i y], y]$ for $i \geq 1$. The word

$$[x, {}_n y] = [x, y, \ldots, y]$$

is called the nth Engel word.

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On the other hand, due to Rips and Juhasz, the expectation now is that for big n these groups need not be nilpotent.

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Thus, finitely generated residually finite n-Engel groups are nilpotent.

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On the other hand, using the result of Wilson one can show that the n-Engel word is concise in residually finite groups for any n.

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$$w = \gamma_k = [x_1, x_2, \ldots, x_k].$$

Theorem

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In particular, it follows that the word γ_k^n is weakly rational. This step is crucial for the proof of the above theorem.

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We know that there exist words that are not weakly rational.

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Now I would like to talk a little about countable conciseness in profinite groups.

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It was conjectured in a joint work with E. Detomi and M. Morigi that if w is a word and G a profinite group such that the set w-values is countable, then $w(G)$ is finite. The conjecture was confirmed for various words w.

In particular, it was confirmed for multilinear commutator words. On the other hand, we were unable to confirm the conjecture for some "easy" words.

For example, we do not know the answer to the following question.

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Let G be a profinite group in which the set $\{x^3 \mid x \in G\}$ is countable. Does it follow that G^3 is finite?

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THE END.   GRAZIE 1000!