Canonical forms and exchange laws for the symmetric group S_n, and for the hyperoctahedral group B_n.

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The symmetric group S_n

The symmetric group S_n is an $n-1$ generated simply-laced Coxeter group which presentation is:

$$\begin{aligned}\langle s_1, s_2, \ldots, s_{n-1} | s_i^2 &= 1, \\ (s_i \cdot s_{i+1})^3 &= 1, \\ (s_i \cdot s_j)^2 &= 1 \quad for \quad |i-j| \geq 2 \rangle.\end{aligned}$$

- The Coxeter generator s_i can be considered as the permutation on n elements which exchange the element i with the element $i+1$, i.e., the transposition $(i, i+1)$;

- We consider multiplication of permutations in left to right order. i.e., for every $\pi_1, \pi_2 \in S_n$, $\pi_1 \cdot \pi_2(i) = \pi_2(j)$, where $\pi_1(i) = j$;

- For every permutation $\pi \in S_n$, the Coxeter length $\ell(\pi)$ is the number of inversions in π, i.e., the number of different pairs i, j, s. t. $i < j$ and $\pi(i) > \pi(j)$;

A canonical form of S_n

$t_2 = s_1$
$t_3 = s_1 \cdot s_2$
$\ldots$
$\ldots$
$\ldots$
$t_n = s_1 \cdot s_2 \cdots s_{n-1}$.

Every element of S_n has a unique presentation in the following canonical form:

$$t_2^{k_2} \cdot t_3^{k_3} \cdots t_{n-1}^{k_{n-1}} \cdot t_n^{k_n}.$$

where, $0 \leq k_j < j$.

The connection between the canonical form and the Coxeter length of the elements

Let $\pi \in S_n$, where $\pi = t_{k_1}^{i_{k_1}} \cdot t_{k_2}^{i_{k_2}} \cdots t_{k_m}^{i_{k_m}}$ in the canonical form s.t. $\sum_{j=1}^{m} i_{k_j} \leq k_1$, then

$$\ell(\pi) = \sum_{j=1}^{m} k_j \cdot i_{k_j} - (i_{k_1} + i_{k_2} + \cdots + i_{k_m})^2.$$

$$\ell(s_r \cdot \pi) = \ell(\pi) - 1, \quad for \quad r = \sum_{j=1}^{m} i_{k_j}.$$

$$\ell(s_r \cdot \pi) = \ell(\pi) + 1, \quad for \quad r \neq \sum_{j=1}^{m} i_{k_j}.$$

In particular, for every two sub-words π_1 and π_2 of π, such that $\pi = \pi_1 \cdot \pi_2$, it is satisfied:

$$\ell(\pi) = \ell(\pi_1 \cdot \pi_2) < \ell(\pi_1) + \ell(\pi_2).$$

Let $\pi, \pi' \in S_n$, where $\pi = t_{k_1}^{i_{k_1}} \cdot t_{k_2}^{i_{k_2}} \cdots t_{k_m}^{i_{k_m}}$, and $\pi' = t_{h_1}^{i_{h_1}} \cdot t_{h_2}^{i_{h_2}} \cdots t_{h_v}^{i_{h_v}}$ in the canonical form, s.t. $h_v \leq k_1$.

If $h_v \leq \sum_{j=1}^{m} i_{k_j} \leq k_1$, then:

$$\ell(\pi' \cdot \pi) = \ell(\pi') + \ell(\pi).$$

Let $\pi \in S_n$, where $\pi = t_{k_1}^{i_{k_1}} \cdot t_{k_2}^{i_{k_2}} \cdots t_{k_m}^{i_{k_m}}$ is written in the canonical form.
Let $\pi_1, \pi_2, \ldots \pi_z$, sub-words of π s.t.

- $\pi_v = t_{h_{v,1}}^{i_{h_{v,1}}} \cdot t_{h_{v,2}}^{i_{h_{v,2}}} \cdots t_{h_{v,m_v}}^{i_{h_{v,m_v}}}$ in the canonical form for every $1 \leq v \leq z$;

- $h_{v-1,m_{v-1}} \leq h_{v,1}$, for every $2 \leq v \leq z$;

- $h_{v-1,m_{v-1}} \leq \sum_{j=1}^{m_v} i_{h_{v,j}} \leq h_{v_1}$;

- $\pi = \pi_1 \cdot \pi_2 \cdots \pi_z$;

Then:

$$\ell(\pi) = \sum_{v=1}^{z} \ell(\pi_v)$$

$$\ell(s_r \cdot \pi) = \begin{cases} \ell(\pi) - 1 & r = \sum_{j=1}^{m_v} i_{h_{v,j}} \\ \ell(\pi) + 1 & otherwise. \end{cases}$$

The exchange laws for the canonical form of S_n

For transforming the element $t_q^{r_q} \cdot t_p^{r_p}$ $(p < q)$ onto the canonical form
$t_2^{i_2} \cdot t_3^{i_3} \cdots t_n^{i_n}$, one need to use the following exchange laws:

$$t_q^{r_q} \cdot t_p^{r_p} = \begin{cases} t_{r_q+r_p}^{r_q} \cdot t_{p+r_q}^{r_p} \cdot t_q^{r_q} & q - r_q \geq p \\ t_{r_q}^{p+r_q-q} \cdot t_{r_q+r_p}^{q-p} \cdot t_q^{r_q+r_p} & r_p \leq q - r_q \leq p \\ t_{p+r_q-q}^{r_q+r_p-q} \cdot t_{r_q}^{p-r_p} \cdot t_q^{r_q+r_p-p} & q - r_q \leq r_p \end{cases}$$

From the described exchange laws for S_n we conclude the following: The canonical form of both elements $t_q^{r_q} \cdot t_p^{r_p}$, where $p < q$, is a product of non-zero powers of at most three different canonical generators, where:

The canonical form of $t_q^{r_q} \cdot t_p^{r_p}$ $(p < q)$ is a product of non-zero powers of two different canonical generators if and only if $q - r_q = p$ or $q - r_q = r_p$, and then the following holds:

$$t_q^{r_q} \cdot t_p^{r_p} = \begin{cases} t_{r_q+r_p}^{r_q} \cdot t_q^{r_q+r_p} & q - r_q = p \\ \\ t_{r_q}^{p-r_p} \cdot t_q^{q-p} & q - r_q = r_p \end{cases}$$

The canonical form of the element $t_q^{r_q} \cdot t_p^{r_p}$ for $q > p$ is either $t_a^{i_a} \cdot t_b^{i_b} \cdot t_c^{i_c}$, or $t_a^{i_a} \cdot t_b^{i_b}$ for $a < b < c$, where $c = q$ and all of the numbers: a, b, i_a, i_b, and i_c, are linear combination of three numbers from $\{p, q, r_p, r_q\}$ with co-efficients 1 or -1

Example

$x = t_2 \cdot t_3^2 \cdot t_4 \cdot t_5^4 = (1,3,5)$.

The permutation x can be written as $x = x_1 \cdot x_2 \cdot x_3$, where:

$x_1 = t_2 = s_1$, $x_2 = t_3^2 \cdot t_4 = s_3$, and
$x_3 = t_5^4 = s_4 \cdot s_3 \cdot s_2 \cdot s_1$.

The reduced form of x in Coxeter generators is just multiplying the reduced forms of x_1, of x_2 and of x_3.

$x = s_1 \cdot s_3 \cdot s_4 \cdot s_3 \cdot s_2 \cdot s_1$,

$$\ell(x) = 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 1 + 5 \cdot 4 - \\ - (4^2 + (1+2)^2 + 1^2) = 6.$$

$y = t_2 \cdot t_3 \cdot t_4^2 \cdot t_5^2 = (2, 4, 5, 3).$

The permutation y can be written as $y = y_1 \cdot y_2$, where:

$y_1 = t_2 \cdot t_3 = s_2$ and $y_2 = t_4^2 \cdot t_5^2 = s_4 \cdot s_3$.

The reduced form of y in Coxeter generators is just multiplying the reduced forms of y_1 and of y_2.

$y = s_2 \cdot s_4 \cdot s_3,$

$$\ell(y) = 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 2 + 5 \cdot 2 - \\ - ((2+2)^2 + (1+1)^2) = 3.$$

Now, we find $x \cdot y$, where we use the described exchange laws.

$$\begin{aligned}
x \cdot y &= t_2 \cdot t_3^2 \cdot t_4 \cdot (t_5^4 \cdot t_2) \cdot t_3 \cdot t_4^2 \cdot t_5^2 \\
&= t_2 \cdot t_3^2 \cdot t_4^2 \cdot (t_5^3 \cdot t_3) \cdot t_4^2 \cdot t_5^2 \\
&= t_2 \cdot t_3^2 \cdot t_4^2 \cdot t_3 \cdot t_4^2 \cdot (t_5^4 \cdot t_4^2) \cdot t_5^2 \\
&= t_2 \cdot t_3^2 \cdot t_4^2 \cdot t_3 \cdot (t_4^2 \cdot t_3) \cdot t_4^2 \cdot t_5^4 \\
&= t_2 \cdot t_3^2 \cdot t_4^2 \cdot (t_3 \cdot t_2) \cdot t_3 \cdot t_4 \cdot t_5^4 \\
&= t_2 \cdot t_3^2 \cdot t_4^2 \cdot t_2 \cdot (t_3^2 \cdot t_3) \cdot t_4 \cdot t_5^4 \\
&= t_2 \cdot t_3^2 \cdot (t_4^2 \cdot t_2) \cdot t_4 \cdot t_5^4 \\
&= t_2 \cdot (t_3^2 \cdot t_3^2) \cdot (t_4^3 \cdot t_4) \cdot t_5 \\
&= t_2 \cdot t_3 \cdot t_5^4.
\end{aligned}$$

The hyperoctahedral group B_n

The hyperoctahedral group B_n is an n generated finite Coxeter group which presentation is:

$$\langle s_0, s_1, \ldots, s_{n-1} | s_i^2 = 1, \\ (s_0 \cdot s_1)^4 = 1 \\ (s_i \cdot s_{i+1})^3 = 1 \quad for \quad i \geq 1, \\ (s_i \cdot s_j)^2 = 1 \quad for \quad |i - j| \geq 2 \rangle.$$

- Each element in the group B_n can be considered as a signed permutation on the set $[\pm n] = \{\pm i | 1 \leq i \leq n\}$, s.t., $\pi(-i) = -\pi(i)$ for every $i \in [\pm n]$;

- s_i for $i \geq 1$ is the signed permutation which exchanges the element i with the element $i+1$ (and exchanges the element $-i$ with the element $-(i+1)$), and s_0 is the signed permutation which exchanges 1 with -1;

- $\ell(\pi) = inv(\pi) - \sum_{i=1}^{n} |\pi(neg(\pi))|$, where $inv(\pi)$ is the number of inversions of π, and $neg(\pi)$ is the set of elements $1 \leq i \leq n$, s.t. $\pi(i) < 0$;

- The parabolic subgroup of B_n, which is generated by $s_1, s_2, \ldots s_n$ (the set elements in B_n, which do not contain s_0 in their reduced form of the presentation in Coxeter generators) is isomorphic to the symmetric group S_n.

A canonical form of B_n

$t_1 = s_0$
$t_2 = s_0 \cdot s_1$
$\dots$
$\dots$
$\dots$
$t_n = s_0 \cdot s_1 \cdots s_{n-1}$.

Every element of S_n has a unique presentation in the following canonical form:

$$t_1^{k_1} \cdot t_2^{k_2} \cdots t_{n-1}^{k_{n-1}} \cdot t_n^{k_n}.$$

where, $0 \leq k_j < 2j$.

Since, $t_j^{2j} = 1$, we may consider unique presentation

where, $-j \leq k_j < j$.

The elements of S_n in B_n

The elements of S_n in the canonical form have the form

$t_{i_1}^{k_{i_1}} \cdot t_{i_2}^{k_{i_2}} \cdots t_{i_m}^{k_{i_m}}$, where:

$-i_j \leq k_{i_j} \leq i_j - 1$, for every $1 \leq j \leq m$,

$k_{i_1} < 0$,

$k_{i_r} \leq \sum_{j=1}^{r} k_{i_j} \leq 0$ for $2 \leq r \leq m-1$, and

$\sum_{j=1}^{m} k_{i_j} = 0$.

$\ell(\pi) = \sum_{j=1}^{m} t_{i_j} \cdot k_{i_j}$.

Example

$$
\begin{aligned}
\pi &= t_3^{-2} \cdot t_4^{-2} \cdot t_5^{3} \cdot t_6 \\
&= (t_3^{-2} \cdot t_4^{2}) \cdot (t_4^{-4} \cdot t_5^{4}) \cdot (t_5^{-1} \cdot t_6) \\
&= s_3 \cdot s_2 \cdot s_4 \cdot s_3 \cdot s_2 \cdot s_1 \cdot s_5.
\end{aligned}
$$

Thus,

$$\ell(\pi) = 3 \cdot (-2) + 4 \cdot (-2) + 5 \cdot 3 + 6 \cdot 1 = 7.$$

Let $\pi \in B_n$, then π has a unique presentation in the following form: $u_1 \cdot v_1 \cdot u_2 \cdot v_2 \cdots u_r$ for some r, s. t. the following holds:

- $u_j \in S_n$, for every $1 \leq j \leq r$;

- $v_j = t_{i_j}^{i_j}$, where for $j_1 > j_2$, $i_{j_1} > i_{j_2}$;

- For every $1 \leq j \leq r$, either $u_j = 1$ or $u_j = t_{i_{j_1}}^{k_{i_{j_1}}} \cdots t_{i_{j_{z_j}}}^{k_{i_{j_{z_j}}}}$ in the canonical form.

  $j_1 \geq j - 1$, for every $2 \leq j \leq r$, and $j_{z_j} \leq j$, for every $1 \leq j \leq r - 1$.

Example

$t_2 \cdot t_3 \cdot t_4^3 \cdot t_5^2 = t_1 \cdot (t_1^{-1} \cdot t_2) \cdot t_2^2 \cdot (t_2^{-2} \cdot t_3 \cdot t_4) \cdot t_4^4 \cdot (t_4^{-2} \cdot t_5^2)$.
$v_1 = t_1$, $v_2 = t_2^2$, $v_3 = t_4^4$,
$u_1 = 1$, $u_2 = t_1^{-1} \cdot t_2$, $u_3 = t_2^{-2} \cdot t_3 \cdot t_4$, and $u_4 = t_4^{-2} \cdot t_5^2$.

Consider the presentation of $\pi \in B_n$ in the form

$u_1 \cdot v_1 \cdot u_2 \cdot v_2 \cdots u_r$ for some r, Then,

$\ell(v_{i-1} \cdot u_i) = \ell(v_{i-1}) + \ell(u_i)$, and

$\ell(v_{i-1} \cdot u_i \cdot v_i) = \ell(v_i) - \ell(u_i) - \ell(v_{i-1})$.

$$\ell(\pi) = u_r + v_{r-1} - u_{r-1} - v_{r-1} + u_{r-2} + v_{r-2} - \cdots \\ \cdots + (-1)^{r-1} u_1.$$

Example

$$\pi = t_2 \cdot t_3 \cdot t_4^3 \cdot t_5^2$$
$$= t_1 \cdot (t_1^{-1} \cdot t_2) \cdot t_2^2 \cdot (t_2^{-2} \cdot t_3 \cdot t_4) \cdot t_4^4 \cdot (t_4^{-2} \cdot t_5^2).$$

Then,

$$\begin{aligned}\ell(\pi) &= \ell(t_4^{-2} \cdot t_5^2) + \ell(t_4^4) - \ell(t_2^{-2} \cdot t_3 \cdot t_4) \\ &- \ell(t_2^2) + \ell(t_1^{-1} \cdot t_2) + \ell(t_1) \\ &= 4 \cdot (-2) + 5 \cdot 2 + 4 \cdot 4 - 2 \cdot (-2) - 3 - 4 \\ &- 2 \cdot 2 + 1 \cdot (-1) + 2 + 1 \\ &= 13.\end{aligned}$$

Indeed, the presentation of π in Coxeter generators is:

$$\pi = s_0 \cdot s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_2 \cdot s_1 \cdot s_0 \cdot s_1 \cdot s_2 \cdot s_3 \cdot s_1 \cdot s_0.$$

The exchange laws for the canonical form of B_n

For transforming the element $t_q^{r_q} \cdot t_p^{r_p}$ $(p < q)$ onto the canonical form $t_1^{i_1} \cdot t_2^{i_2} \cdots t_n^{i_n}$, one need to use the following exchange laws:

The case $r_p \leq p$:

$$t_q^{r_q} \cdot t_p^{r_p} = \begin{cases} t_{r_q}^{r_q} \cdot t_{r_q+r_p}^{r_q} \cdot t_{p+r_q}^{r_p} \cdot t_q^{r_q} \\ \\ t_{r_q}^{2r_q+p-q} \cdot t_{r_q+r_p}^{q-p} \cdot t_q^{r_q+r_p} \\ \\ t_{p+r_q-q}^{r_q+r_p-q} \cdot t_{r_q}^{r_q+p-r_p} \cdot t_q^{q+r_q+r_p-p} \end{cases}$$

The case $r_p > p$:

$$t_q^{r_q} \cdot t_p^{r_p} = \begin{cases} t_{r_q+r_p-p}^{r_q} \cdot t_{p+r_q}^{r_q+r_p} \cdot t_q^{r_q} \\ \\ t_{p+r_q-q}^{p+r_q-q} \cdot t_{r_q}^{p+r_q-q} \cdot t_{r_q+r_p-p}^{q-p} \cdot t_q^{q-p+r_q+r_p} \\ \\ t_{p+r_q-q}^{2r_q+r_p-2q} \cdot t_{r_q}^{2p-r_p} \cdot t_q^{r_q+r_p-2p} \end{cases}$$

The case $r_p = p$:

$$t_q^{r_q} \cdot t_p^{r_p} = \begin{cases} t_{r_q}^{r_q} \cdot t_{p+r_q}^{p+r_q} \cdot t_q^{r_q} & q - r_q \geq p \\ \\ t_{p+r_q-q}^{p+r_q-q} \cdot t_{r_q}^{r_q} \cdot t_q^{q+r_q} & q - r_q < p \end{cases}$$