Left 3-Engel elements in groups

Gunnar Traustason

Department of Mathematical Sciences
University of Bath

Ischia Group Theory 2018

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2. n-Engel groups and elements.
3. Left 3-Engel elements.

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Remark 2. Let G be a group of exponent 8. If

$$a \in L_4(G) \Rightarrow a \in HP(G)$$

then G is locally finite.

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t right 3-Engel $\Rightarrow$ $\langle t \rangle^G$ nilpotent of class ≤ 3. (Newell, 1996)

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TFAE:

(1) Left 3 Engel elements are always contained in the $\mathrm{HP}(G)$.
(2) Every finitely generated sandwich group is nilpotent.

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Corollary. Let G be any group and t a left 3-Engel element of order not divisble by 8. Then $t \in HP(G)$.

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This research was supported by a grant from EPSRC