Steinberg-like characters for non-abelain simple groups

A.E. Zalesski

Ischia Group theory
Ischia, Italy, March 2018

Introduction

Let G be a finite group, $|G|$ the order of G and p a prime dividing $|G|$. Let S be a Sylow p-subgroups of G. We define a Steinberg-like character to be a proper character, possibly reducible, of degree $|S|$ and vanishing at all p-singular elements of G. This is a generalization of the notion of the Steinberg character for a simple group of Lie type in defining characteristic p; the Steinberg character is irreducible and plays a prominent role in the character theory of finite groups of Lie type. So our generalization of the Steinberg character seems to be of certain interest.

Another motivation to study Steinberg-like characters comes from the theory of projective modules.

Let F be an algebraically closed field of char $p > 0$. Projective indecomposable FG-modules (PIM) are exactly indecomposable direct summands of the regular FG-module. These were introduced by Brauer and Nesbitt in 1940 and remain important objects of study in representation theory of finite groups. However, there are very poor information on their dimensions. The only well known fact is that the dimension is a multiple of $|S|$.

This naturally leads to

Problem 1. For a finite group G and a prime p, determine the projective FG-modules of dimension $|S|$.

Explicitly, Problem 1 was first stated by Malle and Wiegel (2008).

Note that every PIM Φ lifts to characteristic 0, in the sense that there is a projective indecomposable K_pG-module $\overline{\Phi}$ whose reduction modulo p yields Φ. Here K_p is the ring of integers of some finite extension K of the field of p-adic numbers. The character of $\overline{\Phi}$ vanishes at the p-singular elements of G.

Clearly, $\overline{\Phi}$ viewed as a KG-module is a direct sum of irreducible KG-modules. The trivial KG-module 1_G occurs in a unique PIM $\overline{\Phi_1}$, which is called the **principal** PIM. Malle and Wiegel (2008) have determined simple groups G whose principal PIM is of dimension $|S|$. It became clear from their work that Problem 1 is non-trivial.

Next step is done in my paper J. Alg. 2013 for G to be an arbitrary simple group of Lie type in defining characteristic of p. No extra example occurred, apart from the well known Steinberg modules.

An experience obtained from this work leaded to a hint that the same phenomenon holds not only for projective but also for arbitrary characters vanishing at the p-singular elements.

So I stated, first for myself, the following problem.

Problem 2. Determine Steinberg-like characters of simple groups.

In the case where G is a group of Lie type in defining characteristic p the question is answered in a joint paper with Marco Pellegrini (2016).

For every group of BN-pair rank 1 we discovered a Steinberg-like characters other than the Steinberg one.

We expected that each such character can be a member of a series depending on the rank n of G, but the things turned out to be opposite.

In fact, for groups of rank greater than 5 no such character exists. I already reported on this work at some conferences so I turn to another work (joint with Malle) in which Problem 2 obtained nearly complete solution.

Note that Problem 2 is easy for sporadic groups, so the cases to be considered were the alternating groups and the groups of Lie type in cross characterisic, that is, with p distinct from the defining characteristic of G.

An easy part of the work is to determine irreducible Steinberg-like characters. These are exactly the defect zero characters of degree $|S|$, and much was known about them. So the actual problem was to determine the reducible Steinberg-like characters. Another easy case is where S is cyclic. We have

Proposition 1. Let G be a non-abelian simple group with *cyclic* Sylow p-subgroup S. If there exists a reducible Steinberg-like characters of G then one of the following holds:
(1) $G = L_2(q)$, $q > 4$ even, $|S| = q + 1$;
(2) $G = L_2(p)$, $|S| = p > 5$;
(3) $G = L_n(q)$, n is an odd prime, $n \not| (q - 1)$, $|S| = (q^n - 1)/(q - 1)$;
(4) $G = U_n(q)$, n is an odd prime, $n \not| (q + 1)$, $|S| = (q^n + 1)/(q + 1)$;
(5) $G = A_p$, $|S| = p \geq 5$;
(6) $G = M_{11}$, $|S| = 11$;
(7) $G = M_{23}$, $|S| = 23$.

For groups of Lie type practically no Steinberg-like character exists. To be precise, we prove

Theorem 1. Let G be a groups of Lie type and p is not the defining characteristic of G. Then G has no reducible Steinberg-like character, unless G is isomorphic to a group of Lie type in defining characteristic p, or $G = PSL_2(q)$, $q+1$ is a p-power, or one of the cases (1) – (4) of Proposition 1 holds.

The alternating groups A_n for $p = 2$ are more difficult, but for $p > 2$ we have

Theorem 2. Let G be an alternating groups and $p > 2$. Then G has no reducible Steinberg-like character, unless $G \cong A_p$ or is isomorphic to a group of Lie type in defining characteristic p.

The case with $p = 2$ in general remains open. However, for $n = 2^k, 2^k + 1$ we have constructed some reducible Steinberg-like characters. We conjecture that all other values of n no Steinberg-like character for $p = 2$ exists.

The following result on projective modules is almost straightforwardly deduced from the above results on Steinberg-like characters.

Theorem 3. Let G be a non-abelian simple group. Let M be a *reducible* projective FG-module of dimension $|S|$. Then either Sylow p-subgroups of G are cyclic and (G, p) are as in cases (1), (3), (5), (7) of Proposition 1, or $G = PSL_2(q)$, $q + 1$ is a 2-power and $p = 2$.

Note that the work with Malle contains results for groups closed to simple such as $PGL_n(q)$, $SL_n(q)$, $GL_n(q)$.

One can try to study the above problems for non-simple groups. However, the following problems seem to be more interesting.

Problem 3. Determine the minimum degree of a projective character of G for every prime dividing $|G|$.

Problem 4. Determine the minimum degree of a character of G vanishing at the p-singular elements (when p divides $|G|$).

Some observations concerning these problems are available in the works mentioned above.

I mention the following result on groups of Lie type (Z, J. Alg. 2013):

Theorem 4. Let $G = PSL_n(p^m)$, $n > 4$, and let χ be a reducible projective character for the prime p. Then $\chi(1) \geq (n-1) \cdot |G|_p$.

This bound is sharp. The existence of a projective module of dimension $n \cdot |G|_p$ is well known, and for $q = 2$ there exists such a module of dimension $(n-1) \cdot |G|_2$.
In addition, there exists a character of degree $(n-1) \cdot |G|_p$ vanishing at the p-singular elements. I expect that this is projective.

In the above paper a similar result is proved also for groups $E_n(p^m)$, $n = 6, 7, 8$, where the bound is shown to be $n \cdot |G|_p$.

Some bibliography

G. Malle and Th. Weigel, Finite groups with minimal 1-PIM, Manuscripta Math. 126(2008), 315 - 332.

G. Malle and A. Zalesski, Steinberg-like characters for finite simple groups, arXiv:1712.08401v2 [math.RT] 16 Jan 2018

M. Pellegrini and A. Zalesski, On characters of Chevalley groups vanishing at the non-semisimple elements, Intern. J. Algebra and Comput. Math. 26(2016), 789 - 841.

A. Zalesski, Low dimensional projective indecomposable modules for Chevalley groups in defining characteristic, J. Algebra 377(2013), 125 – 156.