Generalized Weingarten hypersurfaces embeddedin Euclidian space

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Abstract

In this work we extend the Alexandrov theorem to the compact generalized Weingarten hypersurfaces embedded in Euclidian space, that is an hypersurface whose some of the k^{th} mean curvature H_k are lineary related. ie : for some integers ρ and σ satisfying the inequality $0 \leq s \leq r \leq n-1$; we have :

ασHσ + :::: + αρHρ = β

n is the dimension the hypersurface.

Inroduction :

A classical result by Alexandrov [1] states that a close compact hypersurface with constant mean curvature embedded in Euclidean space must be a round sphere. Replacing the mean curvature H by H_k for κ > 1, and using the Reilly formula, Ros [8] proved that any closed embedded hypersurface in Euclidian space with constant H_k must be a round sphere.

This result was obtained by Montiel and Ros [7] for hypersurfaces with constant H_k in hyperbolic space and hemisphere. Koh [5] and Koh.Lee [6] later generalized Montiel.Ros.s result [7] to hypersurfaces with constant mean curvature ratio ${}^{H_k}/_{H}$.

In a recent work de Lima [2] proved a comparable result for the case of linear Weingarten hypersurfaces. That is an hypersurface satisfying $H_k = aH + b$ for two real constants α > 0 and β > 0.

Following the approach introduced in [4] for the study of generalized Weingarten hypersurfaces in Euclidean space, the Alexandrov theorem to the compact generalized Weingarten hypersurfaces embedded in Euclidian space.

Main results

Theorem 1. Let M^n be a compact generalized Weingarten hypersurface embedded in the Euclidean space with at least a non vanishing k^{th}mean curvature H_k: If either one of the following case holds :

(i) For some integer ρ satisfying the inequality :
$0 \leq r \leq n-1$, the following linear relation holds :

$$a_1 H_1 + :::: + a_r H_r = b \;\ldots\ldots\ldots\ldots (1)$$

with $a_i \geq 0$with at least one non zero, and β > 0.

(ii) For some integer ρ satisfying the inequality :
$0 \leq r \leq n-1$, the following linear relation holds :

$$H_r = a_1 H_1 + :::: + a_{r-1} H_{r-1} \;\ldots\ldots (2)$$

with $a_i \geq 0$with at least one non zero:
Then M^n is a round sphere.

Idea of the proof

Let $\psi : M^n \to \mathbb{R}^{n+1}$ be an ν dimensional compact hypersurface embedding in $\mathbb{R}^{n+1}$. Then M^n is the boundary of a compact domain Ω of $\mathbb{R}^{n+1}, \partial\Omega = M^n$.
Under the hypothesis above there exists at least an elliptic point of M^n. This imply that all H_k are positive functions.
For $1 \leq i \leq n$; the Minkoswki formula is written as ([6]) :

$$\int_M H_{i-1}\, dM + \int_M H_i \langle \psi, N \rangle dM = 0$$

On the other hand, since H is strictly positif and by the inequality (See [7]) :

$$H_{k-1} H_l \geq H_k H_{l-1}$$

we obtain :

$$\sum_{i=1}^{r} \int_M a_i H_{i-1}\, dM \geq b(n+1) vol(\Omega)$$

By applying the divergence theorem, we have :

$$-b \int_M \langle \psi, N \rangle\, dM = b(n+1) vol(\Omega)$$

This imply that all the above inequalities are equals. In particular we obtain :

$$\int_M \frac{1}{H_1} dM = (n+1) vol(\Omega)$$

Wich implies that M^n is a sphere (See [3]).
For *(ii)*, using the Minkowski formula, equation (2) and a reccursive argument, we obtain :

H1= X:H0= X

Were X is a constant depends on $a_1, \ldots, a_r$.
Hence M^n is the round sphere.

References

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