Decomposable solvable multiplication Lie groups and topological loops

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Dedicated to the memory of Karl Strambach

Introduction

The application of Lie theory for topological loops was in the interest of Karl Strambach.

- A set L with a binary operation $(x,y) \mapsto x \cdot y$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and for each $x \in L$ the left translations $\lambda_x : L \to L$, $\lambda_x(y) = x \cdot y$ and the right translations $\rho_x : L \to L$, $\rho_x(y) = y \cdot x$ are bijections of L.
- The multiplication group $Mult(L)$ is the permutation group generated by the left and the right translation maps. The stabilizer of the identity element $e \in L$ in $Mult(L)$ is called the inner mapping group $Inn(L)$ of L.
- The left and right division operations on L are defined by the maps $(x,y) \mapsto x \backslash y = \lambda_x^{-1}(y)$, respectively $(x,y) \mapsto y/x = \rho_x^{-1}(y)$, $x, y \in L$.
- A loop $(L, \cdot)$ is called topological, if L is a topological space and the loop operations $\cdot, \backslash, / : L \times L \to L$ are continuous mappings.
- The kernel of a homomorphism $\alpha : (L, \cdot) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L.
- The centre $Z(L)$ of a loop L consists of all elements z which satisfy the equations $zx = xz, zx \cdot y = z \cdot xy,\ x \cdot yz = xy \cdot z,\ xz \cdot y = x \cdot zy$ for all $x, y \in L$. The loop L is called centrally nilpotent of class n if it has an upper central series of length $n+1$.
- A Lie algebra is called decomposable, if it is the direct sum of two proper ideals.

Objective

Each 2-dimensional connected topological proper loop having a Lie group as its multiplication group has nilpotency class two. This nilpotency property is valid for 3-dimensional connected topological loops L having either a solvable Lie group of dimension at most 5 or a 6-dimensional indecomposable solvable Lie group as their group $Mult(L)$. In this poster we show that the centrally nilpotency of class two property is satisfied for 3-dimensional topological loops L if the group $Mult(L)$ is a 6-dimensional decomposable solvable Lie group.

The 3-dimensional connected simply connected topological loops L having a solvable indecomposable Lie group of dimension at most 6 as their multiplication group are known. To complete this classification for every solvable Lie group of dimension at most 6 we determine the multiplication Lie groups and the inner mapping subgroups for L in the class of the decomposable solvable Lie groups.

The procedures

For the classification of 3-dimensional connected simply connected topological loops L having a solvable Lie group G of dimension 6 as their multiplication group we may proceed in the following way:
As $dim(L) = 3$ first we determine those 3-dimensional Lie subgroups K of G which have no non-trivial normal subgroup of G and satisfy the condition, the normalizer $N_G(K)$ is the direct product $K \times Z$, where Z is the centre of G.
After this we have to find left transversals A and B to K in G such that for all $a \in A$ and $b \in B$ one has $a^{-1}b^{-1}ab \in K$ and G is generated by $A \cup B$.
Since the transversals A and B are continuous, they are determined by 3 continuous real functions of 3 variables.
The condition that the products $a^{-1}b^{-1}ab$, $a \in A$ and $b \in B$, are in K is formulated by functional equations. Finding solution of these functional equations we obtain the classification of those Lie groups which are multiplication groups of loops L.

Results

The 6-dimensional decomposable solvable Lie groups $Mult(L)$ of L have 1- or 2-dimensional centre. Among the 6-dimensional decomposable solvable Lie groups with 1-dimensional centre there are 18 families which are groups $Mult(L)$ of L. The corresponding loops have a centre $Z(L) \cong \mathbb{R}$ such that the factor loop $L/Z(L)$ is isomorphic to $\mathbb{R}^2$. In the class of the 6-dimensional decomposable solvable Lie groups with 2-dimensional centre 9 families can be represented as the group $Mult(L)$ of L. The loops belonging to these families have a 2-dimensional centre $Z(L)$ isomorphic to $\mathbb{R}^2$ such that the factor loop $L/Z(L)$ is isomorphic to $\mathbb{R}$. Hence all these loops are centrally nilpotent of class 2. Their multiplication groups have 3-dimensional commutator subgroup and their inner mapping groups $Inn(L)$ are abelian. The groups $Mult(L)$ depend on at most two real parameters.

The centre of L has dimension 1 or 2

The 6-dimensional decomposable solvable Lie algebras with trivial centre are not the Lie algebra of the multiplication group of a topological loop of dimension 3.

The centre of L has dimension 1

Let L be a 3-dimensional connected simply connected topological proper loop having a solvable decomposable Lie group of dimension 6 with 1-dimensional centre as its multiplication group. Then for the Lie algebras of the groups $Mult(L)$ of L we obtain: $\mathbf{g}_1 = \mathbb{R} \oplus \mathbf{g}_{5,19}^{\alpha=0}$, $\mathbf{g}_2 = \mathbb{R} \oplus \mathbf{g}_{5,20}^{\alpha=0}$, $\mathbf{g}_3 = \mathbb{R} \oplus \mathbf{g}_{5,27}$, $\mathbf{g}_4 = \mathbb{R} \oplus \mathbf{g}_{5,28}^{\alpha=0}$, $\mathbf{g}_5 = \mathbb{R} \oplus \mathbf{g}_{5,32}^{h}$, $\mathbf{g}_6 = \mathbb{R} \oplus \mathbf{g}_{5,33}^{\beta,\gamma}$, $\mathbf{g}_7 = \mathbb{R} \oplus \mathbf{g}_{5,34}^{\alpha}$, $\mathbf{g}_8 = \mathbb{R} \oplus \mathbf{g}_{5,35}^{h,\alpha}$, $\mathbf{g}_9 = \mathfrak{l}_2 \oplus \mathbf{g}_{4,1}$, $\mathbf{g}_{10} = \mathfrak{l}_2 \oplus \mathbf{g}_{4,2}^{\alpha}$, $\mathbf{g}_{11} = \mathfrak{f}_3 \oplus \mathbf{g}_{3,2}$, $\mathbf{g}_{12} = \mathfrak{f}_3 \oplus \mathbf{g}_{3,3}$, $\mathbf{g}_{13} = \mathfrak{f}_3 \oplus \mathbf{g}_{3,4}$, $\mathbf{g}_{14} = \mathfrak{f}_3 \oplus \mathbf{g}_{3,5}$, $\mathbf{g}_{15} = \mathfrak{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,2}$, $\mathbf{g}_{16} = \mathfrak{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,3}$, $\mathbf{g}_{17} = \mathfrak{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,4}$, $\mathbf{g}_{18} = \mathfrak{l}_2 \oplus \mathbb{R} \oplus \mathbf{g}_{3,5}$. Where $\mathfrak{l}_2$ is the 2-dimensional non-abelian Lie algebra and $\mathfrak{f}_3$ is the 3-dimensional nilpotent Lie algebra.

The centre of L has dimension 2

Let L be a 3-dimensional connected simply connected topological proper loop having a solvable decomposable Lie group of dimension 6 with 2-dimensional centre as its multiplication group. Then the following Lie groups are the groups $Mult(L)$ of L:
The nilpotent Lie group $\mathbb{R} \times \mathcal{F}_5$ where $\mathcal{F}_5$ is the 5-dimensional elementary filiform Lie group.
If $Mult(L)$ is solvable, non-nilpotent, then its Lie algebra is one of the following solvable Lie algebras: $\mathbf{g}_1 = \mathbb{R}^2 \oplus \mathbf{g}_{4,2}^{\alpha \neq 0}$, $\mathbf{g}_2 = \mathbb{R}^2 \oplus \mathbf{g}_{4,4}$, $\mathbf{g}_3 = \mathbb{R}^2 \oplus \mathbf{g}_{4,5}^{-1 \leq \gamma \leq \beta \leq 1, \gamma\beta \neq 0}$, $\mathbf{g}_4 = \mathbb{R}^2 \oplus \mathbf{g}_{4,6}^{p \geq 0, \alpha \neq 0}$, $\mathbf{g}_5 = \mathbb{R} \oplus \mathbf{g}_{5,8}^{0<|\gamma|\leq 1}$, $\mathbf{g}_6 = \mathbb{R} \oplus \mathbf{g}_{5,10}$, $\mathbf{g}_7 = \mathbb{R} \oplus \mathbf{g}_{5,14}^{p \neq 0}$, $\mathbf{g}_8 = \mathbb{R} \oplus \mathbf{g}_{5,15}^{\gamma=0}$.

References

A. Al-Abayechi and Á. Figula, Topological loops having solvable decomposable multiplication group, submitted to *Results in Mathematics*, 2020.