Isomorphisms of fusion subcategories on permutation algebras

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Abstract

We present two isomorphisms of categories involving generalized Brauer pairs on p-permutation N-interior G-algebras, extending the results of [4]

Generalized Brauer pairs and p-permutation N-interior G-algebras

Consider the field k of characteristic p and A, a p-permutation N-interior G-algebra over k, where $N \trianglelefteq G$. Also consider $c \in A^G$, a primitive idempotent of that algebra. (Q, f_Q) is an (A, c, G)- Brauer pair if Q is a p-subgroup of G and f_Q is a block of $A(Q)$ with $\mathrm{Br}_Q(c)f_Q \neq 0$. The next proposition is an extension of [6, Lemma 8.6.4] and of [5, IV, Lemma 3.5].

Proposition 1. *Let (Q, f_Q) denote an (A, c, G)-Brauer pair verifying that f_Q is primitive in $A(Q)^{C_N(Q)}$. Consider a p-subgroup S of G and a subgroup H of G with $C_N(Q) \leq H \leq N_G(Q, f_Q)$, such that $Q \leq S \leq H$. Also let f_S be a block of $A(S)$. Then (S, f_S) is an (A, c, G)- Brauer pair with $(Q, f_Q) \leq (S, f_S)$ if and only if (S, f_S) is an $(A(Q), f_Q, H)$-Brauer pair.*

Equivalences of categories of Brauer pairs

In this section we assume that c is a G-invariant primitive idempotent of A^N. If we fix a (A, c, G)- Brauer pair (Q, f_Q) such that Q is a defect group of c in N, there is a maximal (A, c, G)-Brauer pair (P, f_P) such that $(Q, f_Q) \leq (P, f_P)^x$ for some $x \in N$.

Definition 1. *Let (Q, f_Q) be a pair in which f_Q remains primitive in $A(Q)^{C_N(Q)}$. Denote $H = N_G(Q, f_Q)$ and define the sets:*

$F_1 := \{(R, f_R) | (R, f_R)$ *is a* (A, c, G)*-Brauer pair such that* $(Q, f_Q) \leq (R, f_R)$ *as* (A, c, G)*-Brauer pairs* $\}$

$F_2 := \{(R, f_R) | (R, f_R)$ *is a* $(A(Q), f_Q, H)$*-Brauer pair* $\}$

Proposition 2. *Let S, T be two p-subgroups of G and let f_S, f_T denote blocks of $A(S), A(T)$, respectively. The $(S, f_S) \leq (T, f_T)$ as elements of F_1 if and only if $(S, f_S) \leq (T, f_T)$ as elements of F_2.*

Isomorphisms of categories of Brauer pairs

Corollary 1. *Let (U, f_U) be an (A, c, G)-Brauer pair such that $(Q, f_Q) \leq (U, f_U)$. Then (U, f_U) is a maximal (A, c, G)-Brauer pair if and only if it is a maximal $(A(Q), f_Q, H)$-Brauer pair.*

Definition 2. *For any maximal (A, c, G)-Brauer pair (U, f_U) , $\mathcal{F}_{(U,f_U)}(A, c, G)$ are the categories, called fusion systems, introduced in [5, Definition 2.3]. The similar notation is used for $\mathcal{F}_{(U,f_U)}(A(Q), f_Q, H)$. In this case, if (U, f_U) is maximal with $(Q, f_Q \leq (U, f_U)$, the full subcategory of $\mathcal{F}_{(U,f_U)}(A, c, G)$ whose elements are those of F_1 and morphisms are induced by conjugation with elements of G is denoted by $\mathcal{C}$;*

Theorem 1. *The identity functor from $\mathcal{F}_{(U,f_U)}(A(Q), f_Q, H)$ to $\mathcal{C}$ provides an isomorphism of categories.*

Covering points

In this section A is a p-permutation G-algebra and C is an N-interior G-algebra that is also a direct summand of A as k-modules, for some normal subgroup N of G. For the most general situation we refer to [1].

Definition 3. *Let $j \in C^N$ be a primitive idempotent with defect group Q in N and assume j is G-invariant. A point $\alpha \in A^G$ (a conjugacy class of a primitive idempotent) covers j if there is $i \in \alpha$ and a point $\epsilon \subseteq A^N$ with defect group Q verifying the conditions:*

$$ji = ij \neq 0,$$

$$if = fi = f \text{ and } jf_1 = f_1 j = f_1 \text{ for some } f \text{ and } f_1 \text{ belonging to } \epsilon.$$

On pairs determined by covering points

We continue with a p-permutation G-algebra A, in which the N-interior G-algebra C is a k-summand, for a normal subgroup N of G. Let c be a G-invariant primitive idempotent of C^N. Let $Q \leq N$ be a defect group of c and consider (Q, f_Q), a (C, c, G)- Brauer pair such that f_Q is primitive in $C(Q)^{C_N(Q)}$, then we may also consider the sets F_1 and F_2.

Proposition 3. *Let $\alpha \subseteq A^G$ be a point covering c. There is a defect group P of α such that $P \cap N = Q$ and a block f_P in $C(P)$ such that $(P, f_P) \in F_1$.*

Isomorphism of fusion subcategories

In this section we keep the assumptions on C, c and f_Q and we further assume that A is G-interior admitting a $G \times G$-stable k-basis. We also assume that A is projective as an $k[N \times 1]$-module and as an $k[1 \times N]$-module. Let $X := \overline{N}_A^{\mathrm{Aut}(Q)}(Q)$ and $Y := \overline{N}_C^{\mathrm{Aut}(Q)}(Q)$ denote the extended Brauer quotients, an $N_G(Q)$-interior and an $N_N(Q)$-interior $N_G(Q)$-algebra, respectively. See [7],[3, Proposition 2.2] and [2, Section 2] for details.

Proposition 4. *Any point of H on X with defect group P that covers f_Q determines a point of G on A with defect group P that covers c. Moreover, there is a block f_P of $C(P)$ such that $(Q, f_Q) \leq (P, f_P)$ as $(C(Q), f_Q, H)$-Brauer pairs and as (C, c, G)-Brauer pairs.*

Let $\mathcal{C}(P, f_P)$ denote the full subcategory of $\mathcal{C}$ consisting of Brauer pairs (T, f_T) such that $(T, f_T) \leq (P, f_P)$. Similarly, let $\mathcal{P}(P, f_P)$ denote the full subcategory of $\mathcal{F}_{(U,f_U)}(A(Q), f_Q, H)$ consisting of Brauer pairs (T, f_T) with $(T, f_T) \leq (P, f_P)$.

Theorem 2. *The category $\mathcal{C}(P, f_P)$ is isomorphic to $\mathcal{P}(P, f_P)$.*

References

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