GROUPS IN WHICH MANY SUBGROUPS ARE COMMENSURABLE WITH NORMAL SUBG

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generalizing/approximating normality

A subgroup H of a group G is said:
(*nn*) **nearly-normal** if $|H^G : H| < \infty$
that is H has finite index in a normal subgroup H^G of G;
(*cf*) **normal-by-finite or core-finite** if $|H : H_G| < \infty$.
that is there is a normal subgroup H_G of G with finite index in H;
(*cn*) **commensurable with a normal subgroup** if there is $N \lhd G$ such that $|HN : (H \cap N)| < \infty$.

- B.H. Neumann, 1955:
all subgroups of the group G are *nn*=nearly normal if and only if G is **finite-by-abelian**.

- J.T.Buckley, J.C.Lennox, B.H.Neumann, H.Smith, J.Wiegold, 1995:
if G is a CF-group, i,e. all subgroups are *cf*=core-finite then G is **abelian-by-finite**, provided G is PILF (Periodic Images are Locally Finite) .

- C.Casolo, U. Dardano, S. Rinauro, 2018:
if G is a CN-group, i.e. all subgroups are *cn*, then G is **finite-by-abelian-by-finite**, provided G is PILF .

For each prime p there is a soluble p-group which is neither abelian-by-finite nor finite-by-abelian.

generalized T-groups

T = normalilty is transitive i.e. subnormal subgroup are normal;
T^* = *nn* is transitive, equiv. subnormal subgroup are nearly normal;
T_* = *cf* is transitive, equiv. subnormal subgroups are core-finite;
$T[*]$ = *cn* is transitive, equiv. each subnormal subgroup is commensurable with a normal subgroup.

Theorem Let G be a finitely generated soluble group.
1) if G is T, then G is finite-or-abelian. (Robinson, 1964)
2) if G is T^*, then G is abelian-by-finite. (Casolo, 1989)
3) if G is T_*, then G is abelian-by-finite (Franciosi, de Giovanni, Newell, 1989)
6) if G is $T[*]$, then G is abelian-by-finite (Dardano,Dikranjan,Rinauro,2018)

Theorem Let G be a subsoluble group.
1) if G is T, then G is metabelian. (Robinson, 1964)
2) if G is T^*, then G is finite-by-metabelian. (Casolo, 1989)
3) if G is T_*, then G is metabelian-by-finite. (Franciosi, de Giovanni, Newell, 1995)
6) if G is $T[*]$, then G is finite-by-metabelian (Dardano, De Mari, see Theorem A or [1])

more generalized T-groups, see [1]

Theorem A [see 1] *Let G be a subsoluble $T[*]$-group. Then G is finite-by-metabelian, hence G is soluble. Moreover, all finitely generated subgroups of G are CF-groups, hence G is locally (abelian-by-finite).*
In particular, also subsoluble T_*-groups are finite-by-metabelian.

On the other hand, there are soluble T_*-groups of arbitrary derived length which are not finite-by-abelian-by-finite.

Concerning the locally nilpotent case, we generalize the results for T_* to the larger class $T[*]$.

Theorem B [see 1] *Let G be a locally nilpotent $T[*]$-group. Then G is hypercentral. Moreover, if G is periodic, then G is a CN-group with a nilpotent subgroup of index at most 2.*
It follows that any radical $T[*]$-group is soluble.

Note that the dihedral group on a Prüfer 2-group is a locally nilpotent T-group which is non-nilpotent.

considering the rank of subgroups, see[3]

A group G is said to have *infinite rank* if there is no integer r such that every finitely generated subgroup of G can be generated by r elements.

M. De Falco, F. de Giovanni, C. Musella, N. Trabelsi (2015) have considered groups with properties T_+ (T^+, resp.), that is groups in which the condition of being nn (resp. cf) is imposed only to subnormal subgroups *with infinite rank*. They showed that *a periodic soluble group of infinite rank G with property T_+ (T^+, resp.) has the full T_* (T^*, resp.) property, provided one of the following holds:*
- the Hirsch-Plotkin radical of G has infinite rank,
- the commutator subgroup G' is nilpotent.

A similar statement is true also for the property cn.

$T[+]$ = subnormal subgroups of infinite rank are commensurable with normal subgroups

Theorem C [see 3] *Let G be a periodic soluble $T[+]$-group whose Hirsch-Plotkin radical has infinite rank. Then G is a $T[*]$-group.*

Corollary *Let G be a periodic soluble $T[+]$-group (resp. T_+-group) of infinite rank such that $\pi(G')$ is finite. Then all subgroups of G are cn (resp.cf)*

Theorem D [see 3] *Let G be a periodic $T[+]$-group of infinite rank with nilpotent commutator subgroup. Then G is a $T[*]$-group.*

Note that the holomorph G group of the additive group A of the rational numbers is $T[+]$. However all proper non-trivial subgroups of A are not cn in G.

groups minimal wrt some properties [2]

The study of groups whose proper subgroups only have a certain property $\mathcal{P}$ is a standard in the theory of groups. The goal of this kind of investigations is to detect out restrictions for the class of so-called *minimal-non-$\mathcal{P}$* groups, possibly under suitable generalized solubility conditions. The case when $\mathcal{P}$ is the property of being *abelian-by-finite* was considered by B.Bruno and R.E.Phillips (1995) who showed that locally graded minimal-non-AF groups are periodic or perfect.

Theorem E [see 2] *Let G be a locally graded group.*
If all proper subgroups of G are finite-by-abelian-by-finite, then G contains a finite normal subgroup N such that all proper subgroups of G/N are abelian-by-finite.
In particular, if G is minimal-non-(finite-by-abelian-by-finite), then G is periodic and finite-by-metabelian.

Theorem F [see 2] *Let G be a locally graded group whose periodic sections are locally finite. If all proper subgroups of G are CN-groups, then G contains a finite normal subgroup N such that all proper subgroups of G/N are CF-groups, hence G is finite-by-abelian-by-finite.*

We may impose the condition to subgroups with infinite rank only:

Theorem E′ [see 2] *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are finite-by-abelian-by-finite. Then all proper subgroups of G are finite-by-abelian-by-finite.*

Theorem F′ [see 2] *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are CN-groups. Then G is finite-by-abelian-by-finite.*

references

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