Hausdorff dimension in profinite groups

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Introduction

Let G be a profinite group and let $\mathcal{S}$ be a filtration series of G, that is, a descending chain of open normal subgroups $G = G_0 \geq G_1 \geq G_2 \geq \cdots$ such that $\cap_{i \geq 1} G_i = 1$. This filtration induces a translation-invariant metric $d^{\mathcal{S}}$ on G defined as

$$d^{\mathcal{S}}(x, y) = \inf\{|G : G_n|^{-1} \mid xy^{-1} \in G_n\},$$

where $x, y \in G$. This metric, in turns, defines the *Hausdorff dimension function* $\mathrm{hdim}_G^{\mathcal{S}}(X)$ for any subset $X \subseteq G$ with respect to the filtration series $\mathcal{S}$.

Explicit formula for the Hausdorff dimension of closed subgroups

Theorem (Y. Barnea, A. Shalev [1]). The Hausdorff dimension of a closed subgroup H of G with respect to $\mathcal{S}$ is given by

$$\mathrm{hdim}_G^{\mathcal{S}}(H) = \liminf_{n \to \infty} \frac{\log |HG_n : G_n|}{\log |G : G_n|} \in [0, 1].$$

The *Hausdorff spectrum* of G with respect to the filtration series $\mathcal{S}$ is

$$\mathrm{hspec}^{\mathcal{S}}(G) = \{\mathrm{hdim}_G^{\mathcal{S}}(H) \mid H \leq_c G\} \subseteq [0, 1].$$

The Hausdorff dimension function, and hence the Hausdorff spectrum, is sensitive to the choice of the filtration $\mathcal{S}$, as shown in the next examples.

Examples

Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$ and $H = \mathbb{Z}_p \oplus \{0\}$. Then:

- If $\mathcal{S} : G = G_0 \geq G_1 \geq \cdots$ such that $G_i = \langle (p^i, 0), (0, p^i) \rangle$ for every $i \geq 0$, then

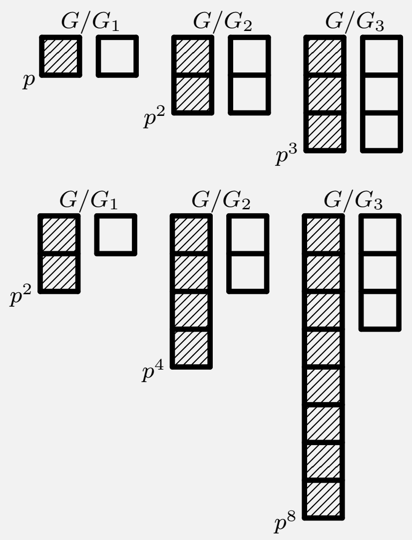
$$\mathrm{hdim}_G^{\mathcal{S}}(H) = 1/2 \quad \text{and} \quad \mathrm{hspec}^{\mathcal{S}}(G) = \{0, 1/2, 1\}$$

  (see the diagram at the top).

- If $\mathcal{S} : G = G_0 \geq G_1 \geq \cdots$ such that $G_i = \langle (p^{2^i}, 0), (0, p^i) \rangle$ for every $i \geq 0$, then

$$\mathrm{hdim}_G^{\mathcal{S}}(H) = 1 \quad \text{and} \quad \mathrm{hspec}^{\mathcal{S}}(G) = \{0, 1\}$$

  (see the diagram at the bottom).



For a finitely generated pro-p group G, however, there are natural choices for $\mathcal{S}$ that encapsulate group-theoretic properties of G. These are:

- The *lower p-series* $\mathcal{L}$: $P_1(G) = G$ and
$$P_i(G) = P_{i-1}(G)^p [P_{i-1}(G), G] \quad \text{for } i \geq 2,$$
- The *dimension subgroup series* $\mathcal{D}$: $D_1(G) = G$ and
$$D_i(G) = D_{\lceil i/p \rceil}(G)^p \prod_{1 \leq j < i} [D_j(G), D_{i-j}(G)] \quad \text{for } i \geq 2$$
- The *p-power series* $\mathcal{P}$:
$$\pi_i(G) = G^{p^i} = \langle g^{p^i} \mid g \in G \rangle \text{ for } i \geq 0.$$
- The *iterated p-power series* $\mathcal{P}^*$: $\pi_0^*(G) = G$ and
$$\pi_i^*(G) = \pi_{i-1}^*(G)^p \text{ for } i \geq 1,$$
- The *Frattini series* $\mathcal{F}$: $\Phi_0(G) = G$ and
$$\Phi_i(G) = \Phi_{i-1}(G)^p [\Phi_{i-1}(G), \Phi_{i-1}(G)] \text{ for } i \geq 1.$$

Some research lines

Normal Hausdorff spectra

Let G be a profinite group and $\mathcal{S}$ a filtration series of G. The *normal Hausdorff spectrum* of G is defined as

$$\mathrm{hspec}_{\trianglelefteq}^{\mathcal{S}}(G) = \{\mathrm{hdim}_G^{\mathcal{S}}(H) \mid H \trianglelefteq_c G\} \subseteq [0, 1].$$

Problem

Does there exist a finitely generated pro-p group G with full normal Hausdorff spectrum with respect to one or several standard filtration series, that is, such that

$$\mathrm{hspec}_{\trianglelefteq}^{\mathcal{S}}(G) = [0, 1]$$

for some $\mathcal{S} \in \{\mathcal{L}, \mathcal{D}, \mathcal{P}, \mathcal{P}^*, \mathcal{F}\}$?

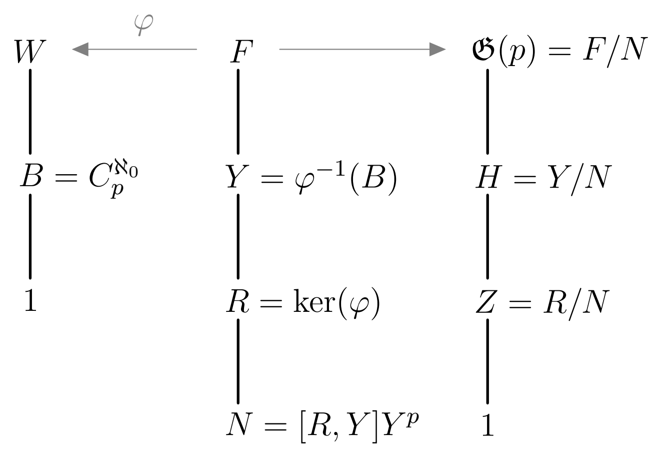
An affirmative answer to the problem

Theorem (IH, B. Klopsch [2]; IH, A. Thillaisundaram [3]). For every prime p, there exists a pro-p group $\mathfrak{G}(p)$ such that

$$\mathrm{hdim}_{\trianglelefteq}^{\mathcal{S}}(\mathfrak{G}(p)) = [0, 1]$$

for every $\mathcal{S} \in \{\mathcal{L}, \mathcal{D}, \mathcal{P}, \mathcal{P}^*, \mathcal{F}\}$.

The construction of such a group, for p odd, is given in the diagram in the right.
In the diagram, $W = C_p \,\hat{\wr}\, \mathbb{Z}_p = \langle \dot{x}, \dot{y} \rangle$ and $F = \langle \tilde{x}, \tilde{y} \rangle$ is the free pro-p group on 2 generators.
Also, φ is the homomorphism from F to W such that $\varphi(\tilde{x}) = \dot{x}$ and $\varphi(\tilde{y}) = \dot{y}$.



Finite Hausdorff spectra (work in progress)

The main problem concerning Hausdorff dimension in the context of profinite groups is characterising p-adic analytic pro-p groups in terms of the finiteness of the Hausdorff spectra. Let us focus on one of the directions of this characterisation, namely, that p-adic analytic implies finite Hausdorff spectra.

Finiteness of the Hausdorff spectra for $\mathcal{D}, \mathcal{P}, \mathcal{P}^*$ and $\mathcal{F}$.

Theorem (Y. Barnea, A. Shalev [1]; B. Klopsch, A. Thillaisundaram, A. Zugadi-Reizabal [4]). Let G be a p-adic analytic pro-p group and H a closed subgroup of G. Then, for $\mathcal{S} \in \{\mathcal{D}, \mathcal{P}, \mathcal{P}^*, \mathcal{F}\}$, we have $\mathrm{hdim}_G^{\mathcal{S}}(H) = \dim(H)/\dim(G)$, where $\dim(H)$ and $\dim(G)$ stand for the analytic dimension of H and G respectively. In particular $\mathrm{hspec}^{\mathcal{S}}(G)$ is finite.

Open problem

Let G be a finitely generated pro-p group and let $\mathcal{S} = \mathcal{L}$. If G is p-adic analytic, does it follow that $\mathrm{hspec}^{\mathcal{S}}(G)$ is finite?

Partial result

Theorem (IH, B. Klopsch). Let G be a p-adic analytic pro-p group and let U be an open uniform subgroup of G. Write L for the free $\mathbb{Z}_p$-Lie algebra (and G-module) associated to U. If the action of G is simple or nilpotent in each indecomposable component of an indecomposable decomposition of L, then $\mathrm{hspec}^{\mathcal{L}}(G)$ is finite. In particular, $\mathrm{hspec}^{\mathcal{L}}(G)$ is finite if one of the following follows:

- G is nilpotent.
- L is a semisimple G-module.

References

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