An approach to skew left braces via triply factorised groups

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Skew left braces

Definition. A *skew left brace* is a triple $(B, +, \cdot)$ with

- $(B, +)$ and $(B, \cdot)$ groups,
- $a \cdot (b + c) = (a \cdot b) - a + (a \cdot c)$ for $a, b, c \in B$.

If, in addition, $(B, +)$ is abelian, we obtain Rump's braces (skew braces of abelian type).

We use ab for $a \cdot b$ and the usual hierarchy for $\cdot$ and $+$.

Skew left braces and derivations

Notation.

- $K = (B, +)$, $C = (B, \cdot)$
- $\lambda\colon C \longrightarrow \operatorname{Aut}(G)$, $\lambda_a(b) = -a + ab$ is a group action.
- The identity map $\delta\colon C \longrightarrow K$ is a derivation or 1-cocycle with respect to λ ($\delta(ce) = \delta(c) + \lambda_c(\delta(e))$ for $c, e \in C$).

Lemma. *Let $\lambda\colon (B, \cdot) \longrightarrow \operatorname{Aut}(A, +)$ be a group action and $\delta\colon (B, \cdot) \longrightarrow (A, +)$ a bijective derivation with respect to λ. Then we can define an addition on B via $b + c = \delta^{-1}(\delta(b) + \delta(c))$ and $(B, +, \cdot)$ becomes a skew left brace.*

Lemma. *Suppose that $\delta\colon C \longrightarrow K$ is a derivation associated to an action λ of C on K and that L is a C-invariant subgroup of K (for instance, this happens when L is a characteristic subgroup of K). Then $\delta^{-1}(L) \leqslant C$.*

Bijective derivations and trifactorised groups

Notation.

- $G = [K]C = K \rtimes C$, with operation $(k_1, c_1)(k_2, c_2) = (k_1 + \lambda_{c_1}(k_2), c_1c_2)$, $k_1, k_2 \in K$, $c_1, c_2 \in C$.
- According to our interest, we consider the elements of B as in K or C.
- We pass from C to K or vice versa by means of the derivation δ.

Lemma. *Let $D = \{(\delta(c), c) \mid c \in C\}$. Then $D \leqslant G$, $G = KD = DC$, and $K \cap D = D \cap C = \{(0, 1)\}$.*

Lemma. *Suppose that $G = [K]C = KD = DC$ and $K \cap D = D \cap C = \{(0, 1)\}$. Then there exists a bijective derivation $\delta\colon C \longrightarrow K$ associated with the action of C on K such that $D = \{(\delta(c), c) \mid c \in C\}$.*

Lemma. *Let $G = [K]C = KD = DC$ with $K \cap D = D \cap C = \{(0, 1)\}$.*

1. *If $L \subseteq K$, then $\delta^{-1}(L) = (-L)D \cap C$.*
2. *If $E \subseteq C$, then $\delta(E) = DE^{-1} \cap K$.*

Relevant commutator relations

We use now multiplicative notation for G. Let $G = [K]C = KD = DC$ with $D \leq G$, $K \cap D = D \cap C = \{1\}$ and let $\delta\colon C \longrightarrow K$ be the corresponding derivation.

Lemma. *Let $k, l \in K$ and $c, e \in C$. Then*

$$\boxed{[kc, le] = \underbrace{\underbrace{[k, e]^c[k, l]^{ec}[c, l]^{c^{-1}ec}}_{\in K}[c, e]}_{} }$$

$$[kc, le] = \underbrace{[k, e]^c[k, l]^{ec}[c, l]^{c^{-1}ec}}_{\in K}\underbrace{[c, e]}_{\in C}.$$

If $k = \delta(c)$, $l = \delta(e)$, then $\boxed{\delta([c, e]) = [k, e]^c[k, l]^{ec}[c, l]^{c^{-1}ec}}$. *In particular, if $H \leq K$, $C \leq \mathrm{N}_G(H)$, and three of the elements $[k, e]$, $[k, l]$, $[c, l]$, $\delta([c, e])$ belong to H, then so does the other one.*

Lemma. *Suppose that $E \leq C$ and that $L = \delta(E)$ is a normal subgroup of G. Then* $\boxed{E \trianglelefteq C \iff [E, C] \subseteq E \iff [K, E] \subseteq L}$.

A "dictionary"

Concept	Skew brace language	Trifactorised group language
left ideal	$\lambda_a(I) \subseteq I$ for all $a \in B$ (equivalently, $B * I \subseteq I$)	$[L, C] \subseteq L$
strong left ideal	$\lambda_a(I) \subseteq I$ for all $a \in B$ and $I \trianglelefteq K$	$[L, G] \subseteq L$ ($L \trianglelefteq G$)
ideal	$a + I = I + a$, $aI = Ia$ for all $a \in B$	$LE \trianglelefteq G$
quotient by an ideal	A/I	$KC/LE \cong [K/L](C/E)$
left nilpotency series	$\mathrm{L}_0(X, Y) = Y$, $\mathrm{L}_n(X, Y) = X * \mathrm{L}_{n-1}(X, Y)$ (for $X = Y = B$, Rump's B^n)	$[[\dots[H, E], \dots], E]$
left π-nilpotent left brace	$\mathrm{L}_n(B, B_\pi) = 0$ for some n	C π-nilpotent (C p-nilpotent for all $p \in \pi$)
socle	$\operatorname{Soc}(B) = \{a \in B \mid ab = a + b,\ b + a = a + b \text{ for all } b \in B\} = \operatorname{Ker}\lambda \cap \mathrm{Z}(B, +)$	$\mathrm{C}_C(K) \cap \delta^{-1}(\mathrm{Z}(K)) = \mathrm{C}_C(K) \cap \mathrm{Z}(K)D \leqslant C$
right nilpotency series	$\mathrm{R}_0(X, Y) = X$, $\mathrm{R}_n(X, Y) = \mathrm{R}_{n-1}(X, Y) * Y$ (for $X = Y = B$, Rump's $B^{(n)}$)	$\delta^{-1}([\cdots[\delta^{-1}([E, H]), H], \dots, H])$
right π-nilpotent left brace	$\mathrm{R}_n(B_\pi, B) = 0$ for some n	$\delta^{-1}([\cdots[\delta^{-1}([E_\pi, K]), K], \dots, K]) = 1$

Factorisations: a Fitting-like ideal

Theorem. *Suppose that a skew left brace of abelian type B can be descomposed as the sum of two ideals that are left nilpotent as left braces. Then B is left nilpotent.*

Theorem. *Let B be a skew left brace of abelian type that can be factorised as the product of an ideal I_1 that is trivial as a left brace and a strong left ideal I_2 that is right nilpotent as a left brace. Then B is right nilpotent.*

Open question. *Is it true that a skew left brace of abelian type that can be factorised as a product of two ideals that are right nilpotent as left braces must be right nilpotent?*

Other results about factorisations are recovered with trifactorised groups.

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