On the algebraic meaning of elliptic functions

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The classic case: $\wp$ as an exp

The present poster collects a couple of remarks that I happened to make in the past years on the Weierstrass elliptic functions $\sigma(t)$, $\zeta(t) = \frac{d}{dt}\ln(\sigma(t))$, and $\wp(t) = \frac{d}{dt}\zeta(t)$, and that, in my opinion, are worth being pointed out to the reader's attention.
I cannot but begin from a milestone, at the crossroad of Algebra, Geometry and Complex analysis, that is, the property that

$$t_1+t_2+t_3=0 \iff \begin{vmatrix} 1 & \wp(t_1) & \wp'(t_1) \\ 1 & \wp(t_2) & \wp'(t_2) \\ 1 & \wp(t_3) & \wp'(t_3) \end{vmatrix} = 0$$

which makes the map $t \mapsto P(t) = (\wp(t), \wp'(t))$ a group homomorphism from the additive group of $\mathbb{C}$ to the Jacobian $\mathfrak{J}(\mathcal{C})$ of the elliptic curve associated to the doubly-periodic Wierstrass elliptic function $\wp$, with period lattice $\Lambda = \langle 1, \widehat{\tau}\rangle$, thus giving in turn a geometric representation of the quotient group $\mathbb{C}/\Lambda$.

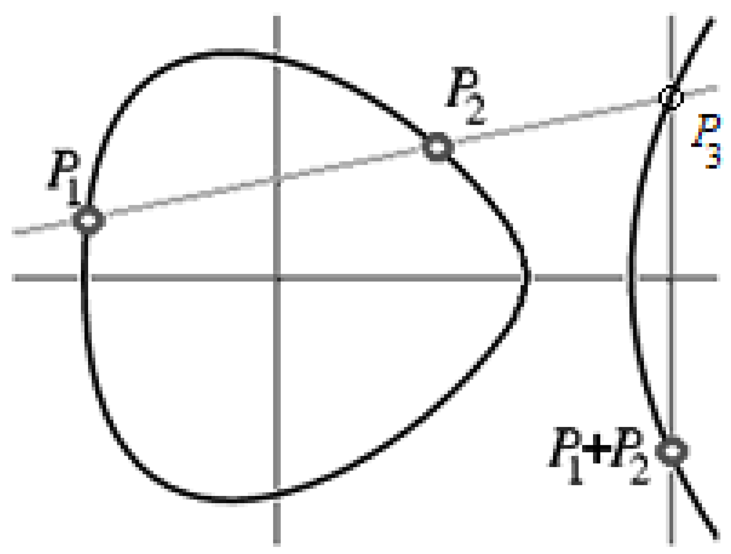

Fig. 1: $P_1+P_2+P_3=0$

The above identity makes the divisor $P_1+P_2-2\Omega$ *linearly equivalent* to the divisor $Q-\Omega$ (where Q is the point marked with P_1+P_2 in the above picture), by means of the function

$$f(x,y) = \frac{\begin{vmatrix} 1 & \wp(t_1) & \wp'(t_1) \\ 1 & \wp(t_2) & \wp'(t_2) \\ 1 & x & y \end{vmatrix}}{\begin{vmatrix} 1 & \wp(t_3) \\ 1 & x \end{vmatrix}}.$$

In fact, we see directly that

$$\operatorname{div}(f) = (P_1+P_2+P_3-3\Omega)-(P_3+Q-2\Omega) = (P_1+P_2)-Q-\Omega,$$

thus

$$(P_1-\Omega)+(P_2-\Omega) = Q-\Omega+\operatorname{div}(f) \equiv Q-\Omega.$$

Toroidal groups: σ as a 2-cocycle

Similarly, a complex periodic function of two variables, whose period lattice $\Lambda = \langle (1,0),(0,1),(\widehat{\tau},\widetilde{\tau})\rangle$ has real rank 3, defines a *toroidal group* $\mathbb{C}^2/\Lambda$. This (non-compact) group is isomorphic to the *generalized Jacobian* $\mathfrak{J}_L$ of an elliptic curve $\mathcal{C}$, where a divisor $L = M+N$ has been fixed, say $M = (\wp(t_M), \wp'(t_M))$ and $N = (\wp(t_N), \wp'(t_N))$, and where now any two divisors are equivalent if they again differ by $\operatorname{div}(f)$, but for a function $f(x,y)$ such that $v_M(1-f)\geq 1$ and $v_N(1-f)\geq 1$, where v_P is the discrete valuation of the local ring $\mathcal{O}_P$ of the rational functions of $\mathcal{C}$ that are regular in P.
The generalized Jacobian $\mathfrak{J}_L$ can be described by the short exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathfrak{J}_L \longrightarrow \mathfrak{J}(\mathcal{C}) \longrightarrow \Omega$$

and by a corresponding (non-regular) 2-cocycle $c_L : \mathfrak{J}(\mathcal{C})\times\mathfrak{J}(\mathcal{C}) \to \mathbb{C}^*$. Note that the non trivial cocycle c_L induces a cocycle $c_LP : \mathbb{C}\times\mathbb{C}\to\mathbb{C}^*$, as well, with $(x_1,x_2)\mapsto c_L\big(P(x_1),P(x_2)\big)$. But, since every commutative Lie group extension of $\mathbb{C}^*$ by $\mathbb{C}$ is splitting, c_LP turns out indeed to be a coboundary $\delta^1(gP) : \mathbb{C}\times\mathbb{C}\to\mathbb{C}^*$, where $gP : \mathbb{C}\to\mathbb{C}^*$ is a section (cf. [1]) defined by

$$gP(t) = \exp(-2\eta_1\widetilde{\tau}t)\frac{\sigma(t_M)}{\sigma(t_N)}\frac{\sigma(t-t_N)}{\sigma(t-t_M)}.$$

How to describe $\delta^1(gP) : \mathbb{C}\times\mathbb{C}\to\mathbb{C}^*$ as $\delta^1(g) : \mathfrak{J}(\mathcal{C})\times\mathfrak{J}(\mathcal{C})\to\mathbb{C}^*$ gives an algebraic meaning to the following two well-known properties of σ:

$$\frac{\sigma(t_1+t_2-t_N)\sigma(t_1+t_N)\sigma(t_2+t_N)\sigma(t_1-t_2)}{\sigma(t_1)^3\sigma(t_2)^3\sigma(-t_N)^3} = -\frac{1}{2}\begin{vmatrix} 1 & \wp(t_1) & \wp'(t_1) \\ 1 & \wp(t_2) & \wp'(t_2) \\ 1 & \wp(-t_N) & \wp'(-t_N) \end{vmatrix},$$

$$\frac{\sigma(t_j+t_N)\sigma(t_j-t_N)}{\sigma(t_j)^2\sigma(t_N)^2} = \begin{vmatrix} 1 & \wp(t_j) \\ 1 & \wp(t_N) \end{vmatrix},$$

combining which, one obtains

$$\frac{\sigma(t_1+t_2-t_N)\sigma(t_N)}{\sigma(t_1-t_N)\sigma(t_2-t_N)}\frac{\sigma(t_1-t_2)}{\sigma(t_1)\sigma(t_2)} = -\frac{1}{2}\frac{\begin{vmatrix} 1 & \wp(t_1) & \wp'(t_1) \\ 1 & \wp(t_2) & \wp'(t_2) \\ 1 & \wp(-t_N) & \wp'(-t_N) \end{vmatrix}}{\begin{vmatrix} 1 & \wp(t_1) \\ 1 & \wp(t_N) \end{vmatrix}\cdot\begin{vmatrix} 1 & \wp(t_2) \\ 1 & \wp(t_N) \end{vmatrix}},$$

and, dividing this by the one with M instead of N, we unveil the coboundary $(c_LP)(t_1,t_2)$ as the cocycle $c_L\big(P(t_1),P(t_2)\big)$

$$(c_LP)(t_1,t_2) = \delta^1(gP)(t_1,t_2) = \frac{\sigma(t_1+t_2-t_N)\sigma(t_N)}{\sigma(t_1-t_M)\sigma(t_2-t_M)}\frac{\sigma(t_1-t_M)\sigma(t_2-t_M)}{\sigma(t_1+t_2-t_M)\sigma(t_M)} =$$

$$\frac{\begin{vmatrix} 1 & \wp(t_1) & \wp'(t_1) \\ 1 & \wp(t_2) & \wp'(t_2) \\ 1 & \wp(-t_N) & \wp'(-t_N) \end{vmatrix}}{\begin{vmatrix} 1 & \wp(t_1) \\ 1 & \wp(t_N) \end{vmatrix}\cdot\begin{vmatrix} 1 & \wp(t_2) \\ 1 & \wp(t_N) \end{vmatrix}}\cdot\frac{\begin{vmatrix} 1 & \wp(t_1) \\ 1 & \wp(t_M) \end{vmatrix}\cdot\begin{vmatrix} 1 & \wp(t_2) \\ 1 & \wp(t_M) \end{vmatrix}}{\begin{vmatrix} 1 & \wp(t_1) & \wp'(t_1) \\ 1 & \wp(t_2) & \wp'(t_2) \\ 1 & \wp(-t_M) & \wp'(-t_M) \end{vmatrix}} = c_L\big(P(t_1),P(t_2)\big).$$

Liftings: ζ as a section

Since ζ is quasi-periodic, there is no way of expressing $\zeta(t)$ in algebraic terms of $\wp(t)$ and $\wp'(t)$. In this direction, the simplest results are probably the well-known addition formulas

$$\zeta(z+w)-\zeta(z)-\zeta(w) = \frac{1}{2}\frac{\wp'(z)-\wp'(w)}{\wp(z)-\wp(w)}, \qquad (1)$$

$$\zeta(2z)-2\zeta(z) = \frac{3\wp^2(z)}{\wp'(z)}-\frac{g_2}{4\wp'(z)}.$$

Notice, however, that the first formula gives exactly $\delta^1(\zeta)$, a fact that, again, we are going to unveil as non-accidental.
Indeed, if $\mathcal{C}$ is defined over $\mathbb{Q}_p$ and has *good reduction* modulo p, that is, the reduction mod p is not singular, then for the group $\mathfrak{J}_{\mathbb{Q}_p}$ we obtain the short exact sequence (cf. [2, Ch. VII])

$$1 \longrightarrow p\mathbb{Z}_p \xrightarrow{\exp} \mathfrak{J}_{\mathbb{Q}_p} \xrightarrow{\text{mod}} \mathfrak{J}_{\mathrm{GF(p)}} \longrightarrow \Omega \qquad (2)$$

Note that $\exp(pt) = (\wp(pt), \wp'(pt)) - \Omega$ is convergent on $p\mathbb{Z}_p$. Writing the points in the reduced curve as

$$P(z) = \operatorname{mod}\exp(z) = \big(\operatorname{mod}\wp(z), \operatorname{mod}\wp'(z)\big),$$

adapting the arguments in [1], one formally finds a section

$$\psi(P(z)) = \int_\Omega^z -\wp(t)\,dt = \zeta(z),$$

that is, $\psi P = \zeta$.
Again, the possibility of expressing the 2-cocycle

$$\Psi\big(P(t_1),P(t_2)\big) = \delta^1(\psi)\,\big(P(t_1),P(t_2)\big)$$

of the extension (2) as $\delta^1(\zeta)\,(t_1,t_2)$ relies on the formulas in (1): for any two given points $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$,

$$\Psi(P_1,P_2) = \delta^1(\psi)\,(P_1,P_2) = \frac{1}{2}\frac{y_1-y_2}{x_1-x_2}$$

and

$$\Psi(P_1,P_1) = \frac{3x_1^2}{y_1}-\frac{g_2}{4y_1}.$$

References

References

[1] Di Bartolo A.; Falcone, G., *The periods of the generalized Jacobian of a complex elliptic curve*, Adv. Geom. (2015)

[2] Silverman J. H., *The arithmetic of elliptic curves*, Springer GTM 106, 1986.