A NOTE ON NON-INNER AUTOMORPHISM CONJECTURE

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Introduction

Let p be a prime, and G be a finite nonabelian p-group. By a celebrated theorem of Gaschütz [7], G admits a non-inner automorphism of p-power order. In 1973, Berkovich [9] proposed that every finite nonabelian p-group has a non-inner automorphism of order p. Using a cohomological result of P. Schmid [11], and [6] every finite nonabelian regular p-group has a non-inner automorphism of order p. In [6] Deaconescu and Silberberg proved the conjecture for groups satisfying the condition $C_G(Z(\Phi(G)) \neq \Phi(G)$. The conjecture was also proved for finite p-groups with $G/Z(G)$ is powerful, and for p-groups of maximal class by Adbollahi [2], for finite p-groups of class 3, and for finite p-groups of coclass 2 by Abdollahi et.al [3], [4], for finite p-groups of coclass 3 with the exception of $p = 3$ by Ruscitti et.al [10], and for odd order p-groups G with $(G, Z(G))$ is a Camina pair by Ghoraishi [8]. In this paper we prove this conjecture for 2-generator finite p-groups $(p \geq 5)$, and as an application we prove this conjecture for finite p-groups of coclass 4 and 5. We achieve this using the notion of Camina triples.

Camina triples

Let $1 < M \leq N$ be two proper normal subgroups of a finite group G. Then (G, N, M) is called a Camina triple if for every $g \in G \setminus N$, $1 \neq m \in M$ $\exists$ $t \in G$ such that $[g, t] = m$, and (G, N, M) is called a Frobenius triple if $C_G(x) \leq N$ for every $1 \neq x \in M$. The following theorem appeared in [5].

Let (G, N, M) be a Camina triple. The following are equivalent: (i) (G, N, M) is a Frobenius triple. (ii) $([G : N], |M|) = 1$. (iii) There exists a subgroup $H \leq G$ such that $G = HN$, $H \cap M = 1$.

Groups with non-cyclic center

Let m be a maximal subgroup of G. Let $z_1, z_2 \in \Omega_1(Z(G))$ such that $\langle z_1\rangle \cap \langle z_2\rangle = 1$, and $g \in G \setminus m$. We assume $Z(G) \lneq Z(m) = C_G(m)$. Then $\exists\alpha_i \in Aut(G)$ of order p such that $\alpha_i(g) = gz_i$, $i = i, 2$. If α_i is inner, $\exists t_i \in Z(m) \cap Z_2(G)$ such that $[g, t_i] = z_i$. Then we show $(G, m, \langle z_i\rangle)$ is a Camina triple. Set $H = \langle g\rangle$, we have $G = Hm$. Noting $(G, m, \langle z_i\rangle)$ is not a Frobenius triple, we get $H \cap \langle z_i\rangle = \langle z_i\rangle$, $i = 1, 2$. This yields $\langle z_1\rangle = \langle z_2\rangle$, a contradiction. This reduces the verification of the conjecture for groups having cyclic center.

Two-generator finite p-groups, $p \geq 5$

We assume $|\Omega_1(Z(G))| = p$, and $|\Omega_1(Z_2(G))| = p^2$ or p^3. The conjecture for 2-generator groups was studied in [1]. Accordingly $Z(\Phi(G)) \lneq Z(G^p\gamma_3(G)) = C_G(G^p\gamma_3(G))$, and

$$|\Omega_1(Z_2(G))|^2 \leq \frac{|Z(G^p\gamma_3(G)) \cap Z_3(G)|}{|Z(G)|} \leq |\Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))|. \quad (1)$$

- We have $|\frac{G}{G^p\gamma_3(G)}| = p^3$, and has a presentation $\langle x, y \mid x^p, y^p, [y, x, x], [y, x, y]\rangle$. Consider $\psi : \Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G)) \times \Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G)) \to \Omega_1(Z(G)) \times \Omega_1(Z(G))$

  $$(a, b) \mapsto ([b, x, x][y, a, x][y, x, a],\ [b, x, y][y, a, y][y, x, b]).$$

  For every $(a, b) \in \ker(\psi)$, we show $a, b \in Z(G^p\gamma_4(G))$, $[y, a][x, b] \in \Omega_1(Z(G))$, and $a, b \in Z(\Phi(G))$.
- Let $|\Omega_1(Z_2(G))| = p^2$. Then $\Omega_1(Z_2(G))$ commutes with y. We deduce $|\ker(\psi)| \leq p^5$. Then $|\Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))|^2 \leq p^7$, contradicts (1).
- Let $[G : G^p\gamma_4(G)] = p^5$, and G has a presentation F/R. Then $R \leq F^p\gamma_4(F)$. Consider $\psi_1 : \Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G)) \to \Omega_1(Z(G)) \times \Omega_1(Z(G))$,

  $$a \mapsto ([y, a, x][y, x, a], [y, a, y]).$$

  Let $a \in \ker(\psi_1)$. By Von dyck's theorem, $\exists\alpha_{1a}|^{x\mapsto xa}_{y\mapsto ya} \in Aut(G)$. We show α_{1a} has order p. If α_{1a} is inner $\forall a \in \ker(\psi_1)$, we obtain $|\Omega_1(Z_2(G))| = p^2$.
- If $[G : G^p\gamma_4(G)] = p^4$, then $\frac{G}{G^p\gamma_4(G)}$ is of maximal class, and has a presentation $\langle s, s_1 \mid s^p, s_1^p, [s_1, s, s_1], [s_1, s, s, s_1], [s_1, s, s, s]\rangle$ [10, Theorem 2.14]. Consider $\psi_2 : \Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G)) \to \Omega_1(Z(G)) \times \Omega_1(Z(G))$

  $$a \mapsto ([s_1, a, s][s_1, s, a], [s_1, a, s_1]).$$

  We show that $\ker(\psi_2) \in \Omega_1(Z_2(G))$, and deduce $|\Omega_1(Z_2(G))| = p^2$.
- Note that $[G^p\gamma_3(G) : G^p\gamma_4(G)] \leq p^2$. If $G^p\gamma_3(G) = G^p\gamma_4(G)$, we have $\gamma_3(G) \leq G^p$. Then G is potent, hence $|\Omega_1(G)| = [G : G^p] = p^3$. Now the conjecture holds for G by (1).

Groups with small cocalss

Let $p \geq 3$, and G be a finite nonabelian p-group of coclass c. If $c \lneq \binom{d(G)+1}{2}$, then G has a non-inner automorphism of order p [12, Proposition 3.3]. Thus if G is of coclass 4 or coclass 5 then the verification of the conjecture reduces to $d(G) = 2$. Hence the conjecture holds for p-groups of coclass 4, 5 when $p \geq 5$.

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