Solutions to the Yang-Baxter equation of pentagonal type

Marzia Mazzotta
Università del Salento - *ISCHIA GROUP THEORY 2020/2021*

Brief introduction

The Yang-Baxter and pentagon equations are two well-known equations of Mathematical Physic and belong to the class of polygon equations [3]. If S is a set, a map $s: S \times S \to S \times S$ is said to be a *set theoretical solution of the quantum Yang-Baxter equation*, or shortly QYBE solution, if

$$s_{23}\, s_{13}\, s_{12} = s_{12}\, s_{13}\, s_{23},$$

where $s_{12} = s \times \mathrm{id}_S$, $s_{23} = \mathrm{id}_S \times s$, and $s_{13} = (\mathrm{id}_S \times \tau)\, s_{12}\, (\mathrm{id}_S \times \tau)$ and τ is the flip map. Instead, s is called a *set-theoretical solution of the pentagon equation*, or shortly PE solution, if

$$s_{23}\, s_{13}\, s_{12} = s_{12}\, s_{23}.$$

Drinfel'd [4] posed the problem of finding all the QYBE solutions.

There are maps satisfying both the equation, thus a description of this kind of maps is given [2]. We call such maps *solutions of the quantum Yang-Baxter equation of pentagonal type*, or briefly, P-QYBE solutions. Furthermore, contrary to what happens in general, we show that among them there are maps whose powers give solutions.

Basics on the pentagon equation

According to the notation introduced in [1], given a set S and a map s from $S \times S$ into itself, we write

$$s(a, b) = (ab,\ \theta_a(b)),$$

where θ_a is a map from S into itself, for every $a \in S$.

Proposition 1 The map $s(a, b) = (ab,\ \theta_a(b))$ is a PE solution on S if and only if the following conditions hold

$$(ab)c = a(bc) \quad (1)$$
$$\theta_a(b)\theta_{ab}(c) = \theta_a(bc) \quad (2)$$
$$\theta_{\theta_a(b)}\theta_{ab} = \theta_b \quad (3)$$

for all $a, b, c \in S$.

Clearly, the condition (1) leads to consider semigroups. For instance, if S is a semigroup and $f \in End(S)$, with $f^2 = f$, then the map $s(a, b) = (ab, f(b))$ is a PE solution on S. In particular, if S is a group, the only invertible PE solution s on S is given by $s(a, b) = (ab, 1)$ (see [5]).
A complete description of not necessarily bijective PE solutions on groups can be found in [1].

P-QYBE solutions

Definition Let S be a set and $s(a, b) = (ab, \theta_a(b))$ a PE solution on S. The map s is said to be a P-QYBE solution if it is also a QYBE solution.

Proposition 2 Let S be a semigroup and $s(a, b) = (ab, \theta_a(b))$ a PE solution on S. Then, the map s is a QYBE solution if and only if

$$abc = a\theta_b(c)bc \quad \text{(Y1)}$$
$$\theta_a\theta_b = \theta_b \quad \text{(Y2)}$$
$$\theta_a(bc) = \theta_{\theta_b(c)}(bc) \quad \text{(Y3)}$$

are satisfied, for all $a, b, c \in S$.

Proposition 3 Let $s(a, b) = (ab, \theta_a(b))$ be a P-QYBE solution on a semigroup S. Then, the following hold:

1. the map θ_a is idempotent, for every $a \in S$;
2. $\theta_{a|_{S^2}} = \theta_{b|_{S^2}}$, for all $a, b \in S$;
3. if $S^2 = S$, then $s(a, b) = (ab, f(b))$, where f is an idempotent endomorphism of S.

Examples

1. The map $s\,(a, b) = (ab, e)$, with e a left identity (or a right identity) for a semigroup S. Note that in the particular case of S a group, the unique P-QYBE solution is $s(a, b) = (ab, 1)$.
2. The map $s\,(a, b) = (ab, b)$, with S is a left quasi-normal semigroup, i.e., $abc = acbc$, for all $a, b, c, \in S$.
3. The map $s\,(a, b) = \left(ab, b^{-1}b\right)$, with S a Clifford semigroup.

We focus on semigroups S belonging to the variety

$$\mathcal{S} := [abc = adbc]$$

which immediately ensures (Y1) (see [6]). In this way, one has to find maps θ_a from S into itself satisfying just (Y2) and (Y3).

Proposition 4
Let $S \in \mathcal{S}$ such that $S^2 = S$. Then, the unique P-QYBE solutions on S are

$$s(a, b) = (ab, f(b)),$$

with f an idempotent endomorphism of S.

Examples

1. If $S \in \mathcal{S}$, the map $s\,(a, b) = (ab, bab)$.
2. The map $s(a, b) = (f(a), g(b))$, where $ab = f(a)$, for all $a, b \in S$.

Powers of P-YBE solutions

We recall that a map s is a QYBE solution if and only if the map $r := \tau s \tau$ is a solution of the braid equation, i.e., it holds

$$r_{12}\, r_{23}\, r_{12} = r_{23}\, r_{12}\, r_{23}.$$

In this case, we say that r is a YBE solution.
Note that if r is a YBE solution, its n-th power r^n is not necessarily a YBE solution.

If s is a P-QYBE solution, we say that the corresponding map $r := \tau s \tau$ is a P-YBE solution.
Theorem 1 Let $S \in \mathcal{S}$ and $r(a, b) = (\theta_a(b), ab)$ a P-YBE solution on S. Then, it holds $r^5 = r^3$ and the powers r^2, r^3, r^4 of the map r are still YBE solutions. In addition, if S is idempotent, it holds $r^4 = r^2$.

There exist P-YBE solutions r for which $r^5 = r^3$, but the powers of r are not solutions. The P-YBE solution $r(a, b) = (b, ab)$ defined on a left quasi-normal semigroup S is such an example.

References

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Contact Information

marzia.mazzotta@unisalento.it