On the Restricted Burnside Problem for Moufang Loops

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The presentation is based on the work with A.Grishkov and E. Zelmanov

Section

Restricted Burnside Problem- RBP
(formulated in the 1930s)
Is it true that for each m, n there are only finitely many finite $m-$generated groups of exponent n?
In the case of the prime exponent p, this problem was positevely proved by A. I. Kostrikin during the 1950s. Using Hall-Higman Reduction Theorem (1956) and the classification of simple groups the case of arbitrary exponent has been completely settled in the affirmative by Efim Zelmanov, [2],[3] who was awarded the Fields Medal in 1994 for his work.

Section

Let L be a set endowed with a binary operation $\cdot$.
$(L,\cdot)$ is a quasigroup if the mappings L_a, R_b are bijections, $\forall a,b \in L$
left multiplication $L_a x = ax$ *right multiplication* $R_b y = yb$
multiplication group $Mult(L)$ - group generated by $\{L_a, R_b\}_{a,b\in L}$
The quasigroup $(L,\cdot,1)$ is a loop if there exists a two-sided neutral element 1.
inner mapping group $Int(L) = \{\phi \in Mult(L) \mid \phi(1) = 1\}$
As common a normal subloop is the kernel of loop homomorphism.
A subloop is normal if and only if it is invariant under inner mappings.
A loop L is called automorphic if $Int(L) \le Aut(L)$ and L is called left automorphic if the mappings $L_{xy}^{-1} \circ L_x \circ L_y$ are automorphisms of L.
nuclei of loop L
$N(L) = \{x \in L \mid (x,a,b) = (a,x,b) = (a,b,x) = 1 \forall a,b, \in L\}$,
where $(ab)c = a(bc)(a,b,c)$, and *center of loop* L
$C(L) = \{x \in N(L) \mid [x,a] = 1 \forall a, \in L\}$, where $ab = ba[a,b]$
By definition abelian group is centrally nilpotent of class 1.
A loop L is centrally nilpotent of class n, if $L/C(L)$ is centrally nilpotent of class $n-1$.
Analogously, one can define the nuclear nilpotency for loop L.
A loop U is called a Moufang loop if it satisfies the following identities:

$$(xy)(zx) = x(yz)x$$

A (nonassociative) algebra is called a Malcev algebra if it satisfies the identities

$$xy = -yx, \quad (xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y$$

Section

As the first result on RBP for Moufang loops one can consider the following
Theorem(R.H. Bruck) (1958)
For commutative Moufang loops the RBP has a positive solution.
More precisely, R.H.Bruck proved, that the index of central nilpotency of any commutative Moufang loop with n generators does not exceed $n-1$.
For Moufang loops of prime exponent the positive solution of RBP was proved by A. Grishkov (1987) (if $p \neq 3$) and G. Nagy (2001) (if $p = 3$).
It is analogue of Theorem of A.I.Kostrikin for groups.
Theorem(P.Plaumann, LS)(2008)
For automorphic Moufang loops the RBP has a positive solution.
This theorem generalizes Bruck Theorem. More general, the class of nuclearly nilpotent loops, which have the positive solution for RBP was described.
Conjecture
For left automorphic Moufang loops the RBP has a positive solution.
Main Theorem (A.Grishkov, LS, E.Zelmanov) (2020)[1]
There is a finite number of finite Moufang loops M of m generators, such that $x^n = 1,\ x \in M,\ n = p^k,\ p$ is a prime > 3.
Sketch of proof:
Moufang loops⇒Groups with triality⇒Lie algebras with triality⇒ Malcev algebras⇒Filippov result + Zelmanov results⇒Main Theorem
A group G with automorphisms ρ and σ is called a group with triality if $\rho^3 = \sigma^2 = (\rho\sigma)^2 = 1$ and

$$[x,\sigma][x,\sigma]^{\rho}[x,\sigma]^{\rho^2} = 1$$

for every $x \in G$, where $[x,\sigma] = x^{-1}x^{\sigma}$. [1]
Let G be a group with triality. Let $U = \{[x,\sigma] | x \in G\}$. Then the subset U endowed with the multiplication

$$a \cdot b = (a^{-1})^{\rho} b (a^{-1})^{\rho^2};\ a, b \in U$$

becomes a Moufang loop.Every Moufang loop U can be obtained in this way from a suitable group with triality, which is finite if U is finite. Moreover, if p is a prime number, then a finite Moufang p-loop can be obtained from a finite p-group with triality
Let $\mathbf{F}_p$ be a field of order p, let G be a group. Zassenhaus filtration of a group G leads to a Lie p-algebra $L_p(G)$.
We call a Lie algebra (resp. Lie p-algebra) L with automorphisms ρ, σ a Lie algebra with triality if $\rho^3 = \sigma^2 = (\rho\sigma)^2 = 1$ and for an arbitrary element $x \in L$ we have

$$(x^{\sigma} - x) + (x^{\sigma} - x)^{\rho} + (x^{\sigma} - x)^{\rho^2} = 0.$$

Section

Lie algebras with triality and Malcev algebras
Lemma
Let G be a group with triality and let p be a prime number. Then $L_p(G)$ is a Lie p-algebra with triality.

Lemma
Let L be a Lie algebra with triality over a field of characteristic $\neq 2, 3$. Let $H = \{x \in L | x^{\sigma} = -x\}$. Then H is a Malcev algebra with multiplication

$$a * b = [a + 2a^{\rho}, b] = [a^{\alpha}, b],$$

where $a, b \in H$, $\alpha = 1 + 2\rho$.
Lemma(V.T. Filippov) *A finitely generated solvable Malcev algebra over a field of characteristic > 3 is nilpotent if and only if each of its Lie homomorphic images is nilpotent.*
Using this lemma and the technique developed by E.Zelmanov [2],[3] we get the **Main Theorem**

Acknowledgements

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References

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[3] E. I. Zelmanov, *Solution of the restricted Burnside problem for 2-groups,* Mat. Sb. 182 (1991), no. 4, 568-592