Covering groups of elementary abelian groups

Dana Saleh[1] and **Dr. Rachel Quinlan**[2]

[1]Zayed University, Dubai
[2]National University of Ireland, Galway

Covering groups

A covering group of an elementary abelian group of order $p^{(n)}$ is a group G of order $p^{n+\binom{n}{2}}$ consisting of the following data:

- G has generators $x_1, \dots, x_n$
- The commutator subgroup of G is equal to the center and is an elementary abelian group of order $p^{\binom{n}{2}}$ or rank $\binom{n}{2}$ generated by $\binom{n}{2}$ simple commutators $[x_i, x_j]$
- $G/Z(G)$ is an elementary abelian group of order $p^{(n)}$, generated by $\bar{x}_1, \dots, \bar{x}_n$, where $\bar{x}$ denotes the coset xZ of Z in G.

Uniform covering group

Definition 1. *A covering group G of C_2^n is uniform if it has a generating set consisting of n elements all having the same square. Such a generating set is called a uniform basis.[1]*

Let G be a uniform covering group of C_2^n with uniform basis $\{x_1, \dots, x_n\}$. Then $x_t^2 = r$ for each t, and r may be represented as a simple graph on vertices labelled by $x_1, \dots, x_n$, where two vertices are adjacent if and only if the corresponding commutator appears in r.

Theorem 1. *The above description determines a bijective correspondence between the isomorphism types of uniform covering groups of C_2^n and the isomorphism types of simple graphs of order n.[2]*

The 2-uniform case

Definition 2. *A covering group of C_2^n is called 2-uniform if it is not uniform, and it possesses a generating set with n elements having two distinct squares.*

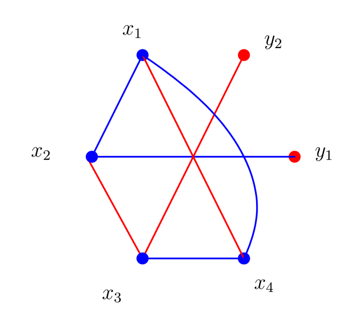
Let G be a 2-uniform covering group of $C_2^{(n)}$. A 2-uniform basis of G is a generating set $\{x_1, \dots, x_k, x_{k+1}, \dots, x_n\}$, where

- x_i in G for $i = 1, \dots, k$: where $x_i^2 = r$, and x_j, for $j = k+1, \dots, n$: where $x_j^2 = s \neq r$, and
- $k \geq \frac{n}{2}$, and
- No element of G is the square of elements from more than k distinct cosets of G' in G.

We may associate graphs (with two colours on both the vertices and edges) with 2-uniform covering groups as in the following example.

Example 1. Let $G = \langle x_1, x_2, x_3, x_4, y_1, y_2 \rangle$, where

- $x_1^2 = x_2^2 = x_3^2 = x_4^2 = r = [x_1, x_2][x_1, x_4][x_2, y_1][x_3, x_4]$,
- $y_1^2 = y_2^2 = s = [x_2, x_3][x_1, x_4][x_3, y_2]$.



Theorem 2. *If $n - k \geq 4$, and neither r nor s is described by a complete subgraph of even order with respect to a 2-uniform basis, then the correspondence between isomorphism types of groups and graphs is one-to-one.*

The situation is more complicated when $n - k \leq 3$, where we may have multiple choices for the element s

Example 2. $n = 7$, $k = 5$, $G = < x_1, x_2, \dots, x_5, x_6, x_7 >$ is a 2-uniform covering group, and $\mathcal{B}_1 = \{x_1, \dots, x_7\}$ is a 2-uniform basis of G where:

- $x_i^2 = [x_5, x_6][x_1, x_4][x_1, x_7][x_4, x_7] = r$ for $i = 1, \dots, 5$
- $x_6^2 = x_7^2 = s = [x_2, x_3][x_1, x_4][x_2, x_4][x_5, x_7]$.
- We write $C_1 = [x_5, x_6]$, corresponding to the complete graph on the vertices x_5 and x_6.
- We write $C_2 = [x_1, x_4][x_1, x_7][x_4, x_7]$, corresponding to the complete graph on x_1, x_4, x_7.

Write $y_6 = x_5 x_6$ and $y_7 = x_1 x_4 x_7$. Then

$$y_6^2 = x_5^2 x_6^2 [x_5, x_6] = r s C_1 = s C_2$$

and

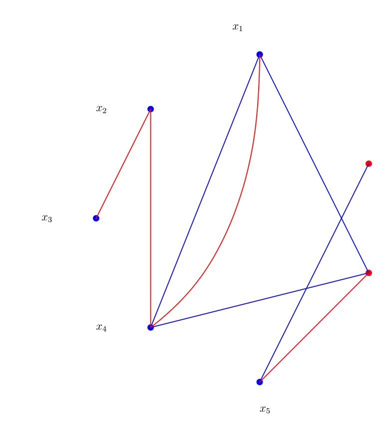
$$y_7^2 = x_1^2 x_4^2 x_7^2 [x_1, x_4][x_1, x_7][x_4, x_7] = s C_2$$

so $y_6^2 = y_7^2 = s' \neq s$ and $\mathcal{B}_2 = \{x_1, \dots, y_6, y_7\}$ is another 2-uniform basis of G.

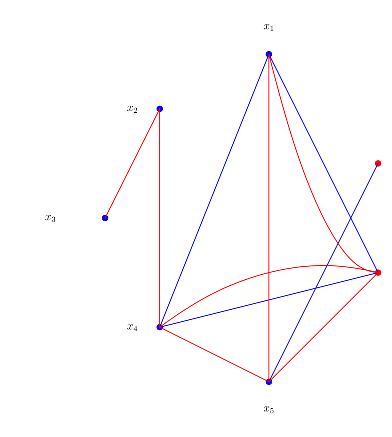
The expression for s' in terms of the basis elements of $\mathcal{B}_2$ is given by

$$\begin{aligned} s' &= s C_2 \\ &= [x_2, x_3][x_1, x_4][x_2, x_4][x_5, x_7][x_1, x_4][x_1, x_7][x_4, x_7] \\ &= [x_2, x_3][x_2, x_4][x_5, x_1 x_4 y_7][x_1, x_1 x_4 y_7][x_4, x_1 x_4 y_7] \\ &= [x_2, x_3][x_2, x_4][x_1, x_5][x_4, x_5][x_5, y_7][x_1, y_7][x_4, y_7]. \end{aligned}$$

The graphs of G with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$ are shown below. They are clearly not isomorphic as graphs, yet they represent the same 2-uniform covering group of C_2^7.



$\Gamma_{\mathcal{B}_1}(G)$



$\Gamma_{\mathcal{B}_2}(G)$

Next steps

The immediate goal is to identify a class of graphs that correspond exactly to the isomorphism classes of 2-uniform covering groups of C_2^n. This task is almost completed but requires careful analysis of special cases such as that in Example 2 above.

References

References

[1] R. Gow and R. Quinlan, *Covering groups of rank 1 of elementary abelian groups*, Common. Algebra, **34(4):1419–1433** (2006).

[2] R. Quinlan, *Real elements and real-valued characters of covering groups of elementary abelian 2-groups*, Common. Algebra, **275(1):191–211** (2004).