Co-prime Graph of Finite Groups

Hayder Baqer Shelash & Mohmad Jasim
Department of Mathematics
Faculty of Comp Sciences and Math
University of Kufa, Iraq

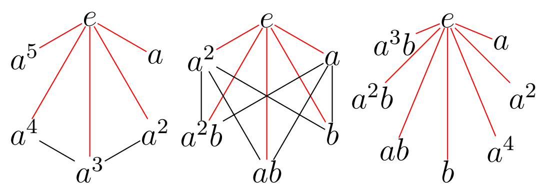
Abstract

Let G be a finite group. The coprime graph Γ_G is define by for any two distinct vertices v_i and v_j of the vertex set are join if and only if $gcd(|v_i|, |v_j|) = 1$. In the paper, we discussed the computing of the number edges of some of the finite group Cyclic group C_n, Dihedral Group D_{2n} and Dicyclic group T_{4n}.

Introduction

Let Γ be a simple graph. The degree of $v \in V(\Gamma)$ is denoted by $deg(v)$. The set of vertices which are adjacent to v is denoted by $N(\Gamma(v))$ and the set of edges denoted by $E(\Gamma) = \{(x, y) \mid gcd(|x|, |y|) = 1\}$.

The Coprime Graph of a finite group G denoted by Γ_G was introduced by Ma et. all in [9] in the year 2014. The graph Γ_G has vertex set as elements of the group G and any two vertices v_i and v_j are adjacent in Γ_G if and only if $gcd(o(v_i), o(v_j)) = 1$ where $o(v_i), o(v_j)$ denote the order of the elements v_i and v_j respectively. They studied various graph theoretic properties of Γ_G namely number of edges and vertices degree sequences of Γ_G. Also several properties of the graph Γ_G when G is the cyclic group and dihedral group when n is prime number was studied by the authors. In [6], Dorbidi extended the results obtained in [7]. For more graphs, see [1], [2], [3], [4], [5] and [8].For example, Fig. 1 is the coprime graph of C_6, D_6 and T_8 denoted by $\Gamma_{C_6}, \Gamma_{D_6}, \Gamma_{T_8}$ respectively . It is easy to see that the coprime graph on G is simple



Main Results

In this section we presented some of the results about co-prime graph of the some of standard groups is define by the following:

$$C_n = \langle a \mid a^n = e \rangle,$$
$$D_{2n} = \langle a, b \mid a^n = b^2 = e, bab = a^{-1} \rangle,$$
$$T_{4n} = \langle a, b \mid a^{2n} = b^4 = e, a^n = b^2, bab^{-1} = a^{-1} \rangle$$

. The vertices degree sequences matrix $\triangle(\Gamma_G)$ is the degree sequence of a graph Γ_G is just a list of the degrees of each vertex in $V(\Gamma_G)$ is define by

$$\triangle(\Gamma_G) = \begin{pmatrix} deg(v_1) & deg(v_2) & \cdots & deg(v_k) \\ \mu(deg(v_1)) & \mu(deg(v_2)) & \cdots & \mu(deg(v_k)) \end{pmatrix}$$

where $deg(v_i)$ is a distinct degree vertices of Γ_G and their $\mu(deg(v_i))$ is the multiplicities for $deg(v_i)$.

Theorem. Let $n = \prod_{i=1}^{s} p_i^{\alpha_i}$ be integer number, the degree sequences matrix of cyclic group is given by the following:

$$\triangle(\Gamma_{C_n}) = \begin{pmatrix} 1 & p_i^{\alpha_i} - 1 & (p_i^{\alpha_i} - 1)(p_j^{\alpha_j} - 1) & \cdots & \prod_{i=1}^{s} p_i^{\alpha_i} - 1 \\ n - 1 & \frac{n}{p_i^{\alpha_i}} & \frac{n}{p_i^{\alpha_i} p_j^{\alpha_j}} & \cdots & \frac{n}{p_i^{\alpha_i} p_j^{\alpha_j} \cdots p_s^{\alpha_s}} \end{pmatrix}$$

Corollary. Let n be a positive integer number, the degree sequences matrix is given by the following:

1. If $n = p^\alpha$, then :
$$\triangle(\Gamma_{C_n}) = \begin{pmatrix} 1 & n - 1 \\ n - 1 & 1 \end{pmatrix};$$
2. If $n = 2^r p^\alpha$, then :
$$\triangle(\Gamma_{C_n}) = \begin{pmatrix} 1 & 2^r - 1 & p^\alpha - 1 & (2^r - 1)(p^\alpha - 1) \\ n - 1 & \frac{n}{2^r} & \frac{n}{p^\alpha} & 1 \end{pmatrix};$$

Theorem.Let $n = p^\alpha$, p be an odd prime number, the degree sequences matrix of co-prime graph of the dihedral group is given by the following:

$$\triangle(\Gamma_{D_{2p^\alpha}}) = \begin{pmatrix} 1 & p^\alpha & p^\alpha - 1 \\ 2p^\alpha - 1 & p^\alpha & p^\alpha + 1 \end{pmatrix}$$

Corollary.The number of all edges when $n = p^\alpha$, $p$odd prime number and $\alpha \geq 1$ of $\Gamma_{D_{2n}}$ be equal to : $E(D_{2p^\alpha}) = n^2 + n - 1$

Theorem.Let $n = \prod_{i=1}^{2} p_i^{\alpha_i}$, p_i be an odd prime number, the degree of sequences is given by the following:

$$\triangle(\Gamma_{D_{2n}}) = \begin{pmatrix} 1 & n & p_1^{\alpha_1} - 1 & p_2^{\alpha_2} - 1 & p_1^{\alpha_1} - 1 p_2^{\alpha_2} - 1 \\ 2n - 1 & n & n + \frac{n}{p_1^{\alpha_1}} & n + \frac{n}{p_2^{\alpha_2}} & n + 1 \end{pmatrix}$$

Corollary.The number of all edges when $n = p^\alpha$, $p$odd prime number and $\alpha \geq 1$ be equal to :

$$E(D_{2p^\alpha}) = n^2 + n - 1$$

Theorem.Let $n = \prod_{i=1}^{s} p_i^{\alpha_i}$ be integer number, the matrix of degree sequences is given by the following:

$$\triangle(\Gamma_{D_{2n}}) = \begin{pmatrix} 1 & n & p_i^{\alpha_i} & p_i^{\alpha_i} p_j^{\alpha_j} & \cdots & p_i^{\alpha_i} p_j^{\alpha_j} \cdots p_s^{\alpha_s} \\ 2n - 1 & n & n + \frac{n}{p_i^{\alpha_i}} & n + \frac{n}{p_i^{\alpha_i}} & \cdots & n + \frac{n}{p_i^{\alpha_i} p_j^{\alpha_j} \cdots p_s^{\alpha_s}} \end{pmatrix}$$

The order elements table of group T_{4n} we can explain by the following table, its clear that for every elements of type $a^i b$ in group T_{4n} has order equal to 4.

Elements	e	a^i $1 \leq i \leq 2n - 1$	$a^i b$ $1 \leq i \leq 2n$
Order	1	$\frac{2n}{gcd(2n,i)}$	4
Number	1	$\varphi(\frac{2n}{i})$	n

Theorem. Let $n = 2^k$, for some $k \in N$ then the co-prime graph $\Gamma(T_{4n})$ is a complete bipartite graph.

$$\Gamma_{T_{2^{r+2}}} = K_{1,2^{r+2}-1}.$$

References

[1] A. Abdollahi, S. Akbari and H. R. Maimani, *Non-commuting graph of a group*, J. Algebra, 298,(2006) , 468–492.

[2] A. Abdollahi and A. M. Hassanabadi, *Non-cyclic graph of a group*, Comm. Algebra, 35 (2007), 2057–2081.

[3] P. Balakrishnan, M. Sattanathan and R. Kala, *The center graph of a group*, South Asian J. Math., 1 (2011), 21–28.

[4] I. Chakrabarty, S. Ghosh and M. Sen, *Undirected power graphs of semigroups*, Semigroup Forum, 78 (2009), 410–426.

[5] C. Gary and P. Zhang, *Introduction to Graph Theory*, Posts and Telecom Press, Beijing, 2006.

[6] Dorbidi, H.R., *A note on the coprime graph of a group*. International Journal of Group Theory, 5(4), 2016, pp.17-22.

[7] Liu, H., Lu, M. and Tian, F., *On the Laplacian spectral radius of a graph*. Linear algebra and its applications, 376(1-3), 2004, pp.135-141.

[8] M. S. Lucido, *Prime graph components of finite almost simple groups*, Rend. Sem. Mat. Univ. Padova, 102 (1999), 1-22.

[9] Ma, X.L., Wei, H.Q. and Yang, L.Y.,*The coprime graph of a group*. International Journal of Group Theory, 3(3), 2014, pp.13-23.