INVERSE SEMI-BRACES AND SOLUTIONS TO THE YANG-BAXTER EQUATION

Paola Stefanelli

Università del Salento - *ISCHIA GROUP THEORY 2020/2021*

Brief introduction

The Yang-Baxter equation is a fundamental equation of statistical mechanics. In [3], Drinfel'd posed the question of finding and classifying all set-theoretical solutions of the Yang-Baxter equation. Given a set S, a map $r : S \times S \to S \times S$ is said to be a *set theoretical solution of the Yang-Baxter equation*, shortly a *solution*, if the relation

$$(r \times \mathrm{id}_S)(\mathrm{id}_S \times r)(r \times \mathrm{id}_S) = (\mathrm{id}_S \times r)(r \times \mathrm{id}_S)(\mathrm{id}_S \times r)$$

is satisfied. Determining all the solutions is still an open problem. One of the most used approach for obtaining solutions is based on *braces*, algebraic structures introduced by Rump [5] that include the Jacobson radical rings. In particular, any brace gives rise to an involutive solution r, i.e., $r^2 = \mathrm{id}$.

The algebraic structure of the *inverse semi-brace* generalizes braces and gives a new research perspective to the problem of finding solutions. Namely, we obtain solutions that are not necessarily bijective, among these new idempotent ones, i.e., solutions r such that $r^2 = r$.

Basics

We recall that a semigroup S is said to be an *inverse semigroup* if, for each $x \in S$, there exists a unique $x^{-1} \in S$ satisfying $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

Definition. [2] Let S be a set with two operations $+$ and $\cdot$ such that $(S, +)$ is a semigroup (not necessarily commutative) and $(S, \cdot)$ is an inverse semigroup. Then, $(S, +, \cdot)$ is *a (left) inverse semi-brace* if

$$a(b + c) = ab + a\left(a^{-1} + c\right) \tag{1}$$

holds, for all $a, b, c \in S$.

Semi-braces [1], [4] and braces [5] are instances of inverse of inverse semi-braces where, in particular, $(S, \cdot)$ is a group.

One can easily obtain examples of inverse semi-braces starting from an arbitrary inverse semigroup.

Example 1. If $(S, \cdot)$ is an inverse semigroup and $(S, +)$ is a right zero semigroup or a left zero semigroup, then S is an inverse semi-brace, which we call *trivial inverse semi-brace*. Clearly, if $|S| > 1$, then such inverse semi-braces are not isomorphic.

Example 2. Let $(S, \cdot)$ be an inverse semigroup and set $a + b = aa^{-1}b$, for all $a, b \in S$. Then, S is an inverse semi-brace. Similarly, the same is true if we consider the opposite sum, i.e., $a + b = bb^{-1}a$, for all $a, b \in S$.

Solutions associated to inverse semi-braces

Let S be an inverse semi-brace, $\lambda : S \to \mathrm{End}(S, +)$, $a \mapsto \lambda_a$ and $\rho : S \to S^S$, $b \mapsto \rho_b$ the maps respectively defined by

$$\lambda_a(b) = a\left(a^{-1} + b\right) \qquad \rho_b(a) = \left(a^{-1} + b\right)^{-1} b,$$

for all $a, b \in S$. Then, we call the map $r_S : S \times S \to S \times S$ given by

$$r_S(a, b) = (\lambda_a(b), \rho_b(a)),$$

for all $a, b \in S$, the *map associated to the left inverse semi-brace S*.

The following are sufficient conditions to obtain solutions through inverse semi-braces.

Theorem 1 [2] Let $(S, +, \cdot)$ be an inverse semi-brace and r_S the map associated to S. If the following are satisfied

1. $(a + b)(a + b)^{-1}(a + bc) = a + bc$
2. $\lambda_a(b)^{-1} + \lambda_{\rho_b(a)}(c) = \lambda_a(b)^{-1} + \lambda_{(a^{-1}+b)^{-1}}\lambda_b(c)$
3. $\rho_b(a)^{-1} + c = \left(b^{-1} + c\right)\left(\rho_{\lambda_b(c)}(a)^{-1} + \rho_c(b)\right)$,

for all $a, b, c \in S$, then the map r_S is a solution.

In general, solutions associated to inverse semi-braces are not bijective.

The previous examples of inverse semi-braces satisfy the conditions of Theorem 1.

Examples **i)** The map r_S associated to S in Example 1 with $(S, +)$ a right zero semigroup is given by

$$r_S(a, b) = \left(ab, b^{-1}b\right)$$

and is an idempotent solution. Similarly, if $(S, +)$ a left zero semigroup, we get the idempotent solution

$$r_S(a, b) = \left(aa^{-1}, ab\right).$$

Note that if $|S| > 1$ such solutions are not isomorphic. In addition, it is clear that the number of inverse semigroups determines a lower bound for idempotent solutions.

ii) The map r_S associated to S in Example 2 with $a + b = aa^{-1}b$ and the map t_S associated to S with $a + b = aa^{-1}b$ are respectively given by

$$r_S(a, b) = \left(ab, ab\,(ab)^{-1}\right) \qquad t_S(a, b) = \left(ab\,(ab)^{-1}, ab\right),$$

and are idempotent solutions. Denoted by τ the twist map, it holds that $t_S = \tau r_S$, hence these two solutions are not isomorphic.

Contact Information

paola.stefanelli@unisalento.it

The double semidirect product of inverse semi-braces

Theorem 2 Let S and T be two inverse semi-braces, $\sigma : T \to \mathrm{Aut}(S)$ a homomorphism from $(T, \cdot)$ into the automorphism group of the inverse semi-brace S, and $\delta : S \to \mathrm{End}(T)$ an anti-homomorphism from $(S, +)$ into the endomorphism semigroup of $(T, +)$. Set ${}^{u}a := \sigma(u)(a)$ and $u^a := \delta(a)(u)$, for all $a \in S$ and $u \in T$, if it holds

$$(uv)^{\lambda_a({}^{u}b)} + u\left(\left(u^{-1}\right)^b + w\right) = u\left(v^b + w\right), \tag{2}$$

then $B := S \times T$ with respect to the operations

$$(a, u) + (b, v) := \left(a + b,\ u^b + v\right) \qquad (a, u)(b, v) := \left(a\,{}^{u}b,\ uv\right),$$

is an inverse semi-brace. We call such an inverse semi-brace B the *double semidirect product* of S and T via σ and δ.

Set $\Omega^a_{u,v} := \left(u^{-1}\right)^a + v$, for all $a \in S$, $u, v \in T$, the map r_B associated to B is given by

$$r_B((a, u), (b, v)) = \left(\left(\lambda_a({}^{u}b),\ u\,\Omega^b_{u,v}\right), \left({}^{\left(\Omega^b_{u,v}\right)^{-1}u^{-1}}\rho_{{}^{u}b}(a),\ \left(\Omega^b_{u,v}\right)^{-1}v\right)\right).$$

Under mild assumptions, the map associated to the double semidirect product of two arbitrary semi-braces is a solution.

Theorem 3 Let S, T be semi-braces and B the double semidirect product of S and T via σ and δ. If r_S and r_T are solutions associated to S and T, respectively, and the following are satisfied

1. $\left(u^1\right)^a = u^a$,
2. $1^a + u = 1 + u$,

for all $a \in S$ and $u \in T$, then the map r_B associated to B is a solution.

References

[1] F. Catino, I. Colazzo, and P. Stefanelli. Semi-braces and the Yang-Baxter equation. *J. Algebra*, 483:163–187, 2017.

[2] F. Catino, M. Mazzotta, and P. Stefanelli. Inverse semi-braces and the Yang-Baxter equation. *J. Algebra*, 573:576–619, 2021.

[3] V. G. Drinfel′ d. On some unsolved problems in quantum group theory. In *Quantum groups (Leningrad, 1990)*, volume 1510 of *Lecture Notes in Math.*, pages 1–8. Springer, Berlin, 1992.

[4] A. Jespers, E.and Van Antwerpen. Left semi-braces and solutions of the Yang-Baxter equation. *Forum Math.*, 31(1):241–263, 2019.

[5] W. Rump. Braces, radical rings, and the quantum Yang-Baxter equation. *J. Algebra*, 307(1):153–170, 2007.