Groups whose proper subgroups of infinite rank are Černikov-by-hypercentral or hypercentral-by-Černikov

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Abstract

It is proved that if G is an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are Černikov-by-hypercentral (resp. hypercentral-by-Černikov), then all proper subgroups of G are Černikov-by-hypercentral (resp. hypercentral-by-Černikov), where $\mathfrak{X}$ is the class defined by N.S. Černikov as the closure of the class of periodic locally graded groups by the closure operations $\acute{\mathrm{P}}$, $\grave{\mathrm{P}}$, R and L.

Introduction

A group G is said to have *finite (Prüfer) rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. If there is no such r, the group G has *infinite rank.* It is not difficult to show that, if G is a locally (soluble-by-finite) group of infinite rank, then it must be rich in subgroups of infinite rank. Moreover, in recent years, many authors studied the structure of locally (soluble-by-finite) groups G of infinite rank in which every proper subgroup of infinite rank belongs to a given class $\mathfrak{Y}$ and they proved that all proper subgroups of G belong to $\mathfrak{Y}$, sometimes the groups G themselves belongs to $\mathfrak{Y}$. In particular, M.R. Dixon, M.J. Evans and H. Smith [2] proved that an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are locally nilpotent is itself locally nilpotent, where $\mathfrak{X}$ is the class introduced in [1] as the class obtained by taking the closure of the class of periodic locally graded groups by the closure operations $\acute{\mathrm{P}}$, $\grave{\mathrm{P}}$, R and L. Clearly $\mathfrak{X}$ is a subclass of the class of locally graded groups that contains all locally (soluble-by-finite) groups. Recall that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. In [1], it is proved that an $\mathfrak{X}$-group of finite rank is almost locally soluble. Using [2] and the fact that locally nilpotent groups of finite rank are hypercentral, one can show that if G is an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are hypercentral has all its proper subgroups hypercentral. The aim is to consider this problem for the classes of Černikov-by-hypercentral and hypercentral-by-Černikov groups.

The Černikov-by-hypercentral case

The main result of this section is the following.

Theorem A ([3]). Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are Černikov-by-hypercentral, then all proper subgroups of G are Černikov-by-hypercentral.

The consideration of the example of Heineken and Mohamed, that is an infinite non-nilpotent group whose proper subgroups are nilpotent and subnormal, shows that a group that satisfies the hypotheses of Theorem A is not in general Černikov-by-hypercentral.

The proof of Theorem A is accomplished through three lemmas.

Lemma A1 ([3]). *Let G be a locally (soluble-by-finite) group of infinite rank. If all proper subgroups of infinite rank of G are Černikov-by-hypercentral, then G is either Černikov-by-hypercentral or locally finite.*

Lemma A2 ([3]). *Let G be a locally finite group of infinite rank whose proper subgroups of infinite rank are Černikov-by-hypercentral. If G has a proper subgroup of finite index, then all proper subgroups of G are Černikov-by-hypercentral.*

Lemma A3 ([3]). *If G is a locally finite group of infinite rank whose proper subgroups of infinite rank are Černikov-by-hypercentral, then G is not simple.*

Sketch of the proof of Theorem A. First note that finitely generated subgroups of G are of finite rank, and hence soluble-by-finite by [1]. It follows that G is locally (soluble-by-finite). By contradiction, suppose that G has a proper subgroup H of finite rank which is not Černikov-by-hypercentral. Lemma A1 implies that G is locally finite. Suppose first that the commutator subgroup G' is properly contained in G. We have proved that all proper normal subgroups of G are hypercentral. In particular, $\langle G', E\rangle$, is hypercentral for all finitely generated subgroups E of G, and we see that G is locally nilpotent. The contradiction follows using the fact that locally nilpotent groups of finite rank are hypercentral. Thus G is a perfect group, and it follows that all proper normal subgroups of G have finite rank. By Lemma A3, G has no simple images. In this case, G will be nilpotent, a final contradiction.

The hypercentral-by-Černikov case

Our second main result is the following.

Theorem B ([3]). *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of G of infinite rank are hypercentral-by-Černikov, then all proper subgroups of G are hypercentral-by-Černikov.*

It is not known whether a locally (soluble-by-finite) group of infinite rank whose proper subgroups are hypercentral-by-Černikov has to be itself hypercentral-by-Černikov.

The proof of Theorem B requires one preliminary result.

Lemma B1 ([3]). *If G is a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are hypercentral-by-Černikov, then G is not simple.*

Sketch of the proof of Theorem B. Assume, for a contradiction, that G has a proper subgroup of finite rank H which is non-hypercentral-by-Černikov. So G has no proper subgroups of finite index. Suppose first that all proper normal subgroups of G are of finite rank. By Lemma B1, G has no simple images, and so G will be nilpotent, which is a contradiction. Thus G has a proper normal subgroup M of infinite rank, and it follows that G is imperfect. We have proved that G is locally nilpotent-by-Chernikov, in particular, H is hypercentral-by-Černikov, our final contradiction.

References

[1] N.S. Černikov, A theorem on groups of finite special rank, *J. Ukrain. Math.* **42** (1990), 855-861.
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