Rationality of representation zeta functions of compact p-adic analytic groups

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Introduction

Let G be a topological group. For $n \in \mathbb{N}$ let

$$\mathrm{Irr}_n(G)$$

be the set of characters of n-dimensional irreducible (continuous) complex representations of G. Set

$$r_n(G) = \#\mathrm{Irr}_n(G)$$

When G is *representation rigid*, i.e. $r_n(G)$ is finite for all $n \in \mathbb{N}$, one studies the arithmetic and asymptotic properties of the sequence $\{r_n(G)\}_{n\in\mathbb{N}}$ by means of the (*representation*) *zeta function* of G, that is

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s} = \sum_{\theta\in \mathrm{Irr}(G)} \theta(1)^{-s} \ (s \in \mathbb{C}).$$

Properties of the sequence of the $r_n(G)$'s correspond to analytic properties of $\zeta_G(s)$. For example, the abscissa of convergence of this Dirichlet series gives the degree of polynomial growth of the sequence of the $r_n(G)$'s and in some cases a linear recurrence relation for the $r_n(G)$'s may be detected from $\zeta_G(s)$.

Linear recurrence relations and rationality

Fix a prime number p. If G is a pro-p group and its open subgroups have finite abelianisation (FAb for short), then G is representation rigid and $\zeta_G(s)$ is a power series in p^{-s}.

Proposition 1 (**[4, Theorem 4.1.1]**) *Let $S(t) = \sum_{n\in\mathbb{N}} a_n t^n$ be a power series in t with complex coefficients. Then a_n satisfies a linear recurrence relation if and only if $S(t)$ is the power series of a rational function in $\mathbb{C}(t)$.*

Jaikin-Zapirain showed that rationality of the zeta function holds for a large class of groups.

Theorem 2 (**[3, Theorem 1.1]**) *Let G be a FAb compact p-adic analytic group G with $p \neq 2$. Then there are natural numbers $n_1, \dots, n_k$ and functions $f_1(p^{-s}), \dots, f_k(p^{-s})$ rational in p^{-s} such that*

$$\zeta_G(s) = \sum_{i=1}^{k} n_i^{-s} f_i(p^{-s}). \tag{1}$$

When G is pro-p, the same result shows that representation zeta function is rational. This, by Proposition 1, implies that in this case the $r_n(G)$'s satisfy a finite linear recurrence relation.

Surprisingly the previous theorem has the consequence that the zeta function of those groups vanish at -2. Namely the following corollary has been proved by González-Sánchez, Jaikin-Zapirain, and Klopsch.

Corollary 3 (**[1, Theorem 1]**) *Let G be an infinite FAb compact p-adic analytic group G with $p \neq 2$. Then $\zeta_G(-2) = 0$.*

Definition 4 *When the zeta function has the form* (1) *we say that it is* virtually rational.

Main Theorem

In [5] we give a new proof of Jaikin-Zapirain's [3, Theorem 1.1] that holds also for $p = 2$.

Theorem 5 *Let G be a FAb compact p-adic analytic group. Then $\zeta_G(s)(s)$ is virtually rational in p^{-s}. If in addition G is pro-p, then $\zeta_G(s)(s)$ is rational in p^{-s}.*

Step 1: Splitting the zeta function

Every Fab compact p-adic analytic group G contains a uniformly powerful open normal subgroup N. We use this fact to split the zeta function.

Let θ be an irreducible character of N and let $K = \mathrm{Stab}_G(\theta)$. Then it is known that induction of characters is one-to-one from K to G if one restricts to irreducibles lying above θ.

$$\begin{array}{ccc} G & \qquad & \mathrm{Ind}_K^G \lambda_1, \dots, \mathrm{Ind}_K^G \lambda_\ell \\ | & & | \\ K = \mathrm{Stab}_G(\theta) & & \lambda_1, \dots, \lambda_\ell \\ | & & | \\ N & & \theta \end{array}$$

Moreover the irreducible characters $\lambda_1, \dots, \lambda_\ell$ are controlled by the cohomology class $\mathcal{C}(K, N, \theta)$ (roughly speaking the factor set of a projective representation of K extending θ).

This way, for each $c \in H^2(K/N, \mathbb{C}^\times)$ and each $N \leq K \leq G$, we find a finite Dirichlet series $f_K^c(s)$ such that

$$\zeta_G(s)(s) = \sum_{N\leq K\leq G} |G : K|^{-s-1} \sum_{c\in H^2(K/N,\mathbb{C}^\times)} f_K^c(s) Z_{N,K}^c \quad ,$$

where $Z_{N,K}^c$ is the zeta function of N where the summation extends only to the irreducible characters θ with stabiliser K and cohomology $\mathcal{C}(K, N, \theta) = c$. The problem is now reduced to showing that $Z_{N,K}^c$ is a rational function in p^{-s}.

Step 2: Linearising the problem

The next step is to linearise the problem using the fact that every irreducible character of N is induced from a finite index subgroup. We developed the following characterisation to make sure that we are able to recognise the cohomology class at the level of the inducing subgroup.

Proposition 6 *Let P be a pro-p group and N a normal subgroup of P. Let (P, N, θ) be a character triple and $c \in H^2(P/N, \mathbb{C}^\times)$. Then $\mathcal{C}(P, N, \theta) = c$ if and only if there exists a subgroup $H \leq P$ and a character triple $(H, N \cap H, \chi)$ where χ is linear character such that*

i) $P = NH$

ii) $\theta = \mathrm{Ind}_{N\cap H}^N \chi$

iii) $\mathcal{C}(H, N \cap H, \chi) = c$.

Step 3: Model Theory

With the linearisation above, the problem of counting characters of N with a given cohomology class is translated to an enumeration of equivalence classes. Indeed, we consider the set $\mathcal{H}$ of pairs (H, χ) where $H \leq N$ has finite index and χ is a linear character of H that induces irreducibly to N. For each $\theta \in \mathrm{Irr}(N)$ we have several choices for H and χ

$$\begin{array}{ccc} & \theta & \\ \swarrow & & \searrow \\ (H_1, \chi_1) & \dots & (H_\ell, \chi_\ell). \end{array}$$

We therefore introduce an equivalence relation

$$(H_1, \chi_1) \sim (H_2, \chi_2) \iff \mathrm{Ind}_{H_1}^N \chi_1 = \mathrm{Ind}_{H_2}^N \chi_2.$$

Now

1. The set $\mathcal{H}$ is defined by first order sentences in the language $\mathcal{L}$ of compact p-adic analytic groups.
2. The equivalence relation is defined by first order sentences in $\mathcal{L}$.
3. The cohomology group is defined by first order sentences in $\mathcal{L}$.
4. For a pair (H, χ), the property that $\mathcal{C}(K, N, \mathrm{Ind}_H^N \chi) = c$ is defined by first order sentences in $\mathcal{L}$.

We finish the proof using a result by Cluckers saying that in this case the Dirichlet series is rational.

Theorem 7 (**[2, Theorem A.2]**) *Let $d \in \mathbb{N}$. Let E be a (first-order) definable family of equivalence relations in $\mathcal{L}$ on a definable family $X \subseteq \mathbb{Q}_p^d \times \mathbb{N}_0$. Suppose that for each $n \in \mathbb{N}_0$ the quotient X_n/E_n is finite, say, of size a_n. Then the series*

$$\sum_{n\in\mathbb{N}_0} a_n t^n$$

is a rational power series in t over $\mathbb{Q}$ whose denominator is a product of factors of the form $(1 - p^i t^j)$ for some integers i, j with $j > 0$.

References

References

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