Michele Zordan¹

joint work with Alexander Stasinski²

¹ Imperial College London, ² Durham University

Imperial College London

Introduction

Let G be a topological group. For $n \in \mathbb{N}$ let

$$\operatorname{Irr}_n(G)$$

be the set of characters of n-dimensional irreducible (continuous) complex representations of G. Set

$$r_n(G) = \# \mathrm{Irr}_n(G)$$

When G is *representation rigid*, i.e. $r_n(G)$ is finite for all $n \in \mathbb{N}$, one studies the arithmetic and asymptotic properties of the sequence $\{r_n(G)\}_{n\in\mathbb{N}}$ by means of the (*representation*) *zeta function* of G, that is

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s} = \sum_{\theta \in \operatorname{Irr}(G)} \theta(1)^{-s} \ (s \in \mathbb{C}).$$

Properties of the sequence of the $r_n(G)$'s correspond to analytic properties of $\zeta_G(s)$. For example, the abscissa of convergence of this Dirichlet series gives the degree of polynomial growth of the sequence of the $r_n(G)$'s and in some cases a linear recurrence relation for the $r_n(G)$'s may be detected from $\zeta_G(s)$.

Linear recurrence relations and rationality

Fix a prime number p. If G is a pro-p group and its open subgroups have finite abelianisation (FAb for short), then G is representation rigid and $\zeta_G(s)$ is a power series in p^{-s} .

Proposition 1 ([4, Theorem 4.1.1]) Let $S(t) = \sum_{n \in \mathbb{N}} a_n t^n$ be a power series in t with complex coefficients. Then a_n satisfies a linear recurrence relation if and only if S(t) is the power series of a rational function in $\mathbb{C}(t)$.

Jaikin-Zapirain showed that rationality of the zeta function holds for a large class of groups.

Theorem 2 ([3, Theorem 1.1]) Let G be a FAb compact p-adic analytic group G with $p \neq 2$. Then there are natural numbers n_1, \ldots, n_k and functions $f_1(p^{-s}), \ldots, f_k(p^{-s})$ rational in p^{-s} such that

$$\zeta_G(s) = \sum_{i=1}^k n_i^{-s} f_i(p^{-s}). \tag{1}$$

When G is pro-p, the same result shows that representation zeta function is rational. This, by Proposition 1, implies that in this case the $r_n(G)$'s satisfy a finite linear recurrence relation.

Surprisingly the previous theorem has the consequence that the zeta function of those groups vanish at -2. Namely the following corollary has been proved by González-Sánchez, Jaikin-Zapirain, and Klopsch.

Corollary 3 ([1, Theorem 1]) Let G be an infinite FAb compact p-adic analytic group G with $p \neq 2$. Then $\zeta_G(-2) = 0$.

Definition 4 When the zeta function has the form (1) we say that it is virtually rational.

Main Theorem

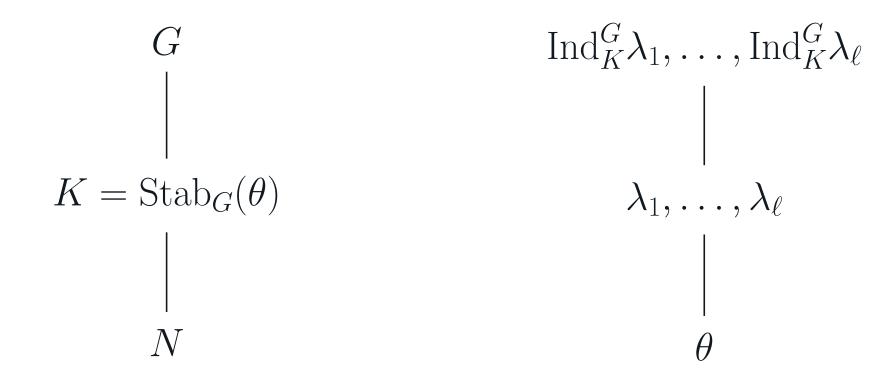
In [5] we give a new proof of Jaikin-Zapirain's [3, Theorem 1.1] that holds also for p=2.

Theorem 5 Let G be a FAb compact p-adic analytic group. Then $\zeta_G(s)(s)$ is virtually rational in p^{-s} . If in addition G is pro-p, then $\zeta_G(s)(s)$ is rational in p^{-s} .

Step 1: Splitting the zeta function

Every Fab compact p-adic analytic group G contains a uniformly powerful open normal subgroup N. We use this fact to split the zeta function.

Let θ be an irreducible character of N and let $K = \operatorname{Stab}_G(\theta)$. Then it is known that induction of characters is one-to-one from K to G if one restricts to irreducibles lying above θ .



Moreover the irreducible characters $\lambda_1, \ldots, \lambda_\ell$ are controlled by the cohomology class $\mathcal{C}(K, N, \theta)$ (roughly speaking the factor set of a projective representation of K extending θ).

This way, for each $c \in H^2(K/N,\mathbb{C}^\times)$ and each $N \leq K \leq G$, we find a finite Dirichlet series $f_K^c(s)$ such that

$$\zeta_G(s)(s) = \sum_{N \le K \le G} |G:K|^{-s-1} \sum_{c \in H^2(K/N,\mathbb{C}^\times)} f_K^c(s) Z_{N,K}^c ,$$

where $Z_{N,K}^c$ is the zeta function of N where the summation extends only to the irreducible characters θ with stabiliser K and cohomology $\mathcal{C}(K,N,\theta)=c$. The problem is now reduced to showing that $Z_{N,K}^c$ is a rational function in p^{-s} .

Step 2: Linearising the problem

The next step is to linearise the problem using the fact that every irreducible character of N is induced from a finite index subgroup. We developed the following characterisation to make sure that we are able to recognise the cohomology class at the level of the inducing subgroup.

Proposition 6 Let P be a pro-p group and N a normal subgroup of P. Let (P,N,θ) be a character triple and $c\in H^2(P/N,\mathbb{C}^\times)$. Then $\mathcal{C}(P,N,\theta)=c$ if and only if there exists a subgroup $H\leq P$ and a character triple $(H,N\cap H,\chi)$ where χ is linear character such that

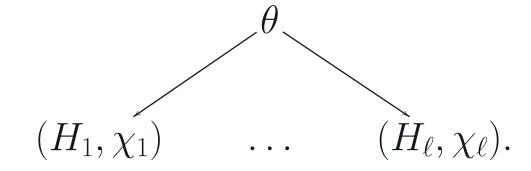
i)
$$P = NH$$

ii)
$$\theta = \operatorname{Ind}_{N \cap H}^{N} \chi$$

iii)
$$C(H, N \cap H, \chi) = c$$
.

Step 3: Model Theory

With the linearisation above, the problem of counting characters of N with a given cohomology class is translated to an enumeration of equivalence classes. Indeed, we consider the set $\mathcal H$ of pairs (H,χ) where $H\leq N$ has finite index and χ is a linear character of H that induces irreducibly to N. For each $\theta\in\mathrm{Irr}(N)$ we have several choices for H and χ



We therefore introduce an equivalence relation

$$(H_1,\chi_1) \sim (H_2,\chi_2) \iff \operatorname{Ind}_{H_1}^N \chi_1 = \operatorname{Ind}_{H_2}^N \chi_2.$$

Now

- 1. The set \mathcal{H} is defined by first order sentences in the language \mathcal{L} of compact p-adic analytic groups.
- 2. The equivalence relation is defined by first order sentences in \mathcal{L} .
- 3. The cohomology group is defined by first order sentences in \mathcal{L} .
- 4. For a pair (H, χ) , the property that $\mathcal{C}(K, N, \operatorname{Ind}_H^N \chi) = c$ is defined by first order sentences in \mathcal{L} .

We finish the proof using a result by Cluckers saying that in this case the Dirichlet series is rational.

Theorem 7 ([2, Theorem A.2]) Let $d \in \mathbb{N}$. Let E be a (first-order) definable family of equivalence relations in \mathcal{L} on a definable family $X \subseteq \mathbb{Q}_p^d \times \mathbb{N}_0$. Suppose that for each $n \in \mathbb{N}_0$ the quotient X_n/E_n is finite, say, of size a_n . Then the series

$$\sum_{n\in\mathbb{N}_0}a_n$$

is a rational power series in t over $\mathbb Q$ whose denominator is a product of factors of the form $(1-p^it^j)$ for some integers i,j with j>0.

References

References

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