Compact groups with a set of positive Haar measure satisfying a nilpotent law

Alireza Abdollahi
University of Isfahan
(Joint work with M. Soleimani Malekan)
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Co-Author

This is a joint work with **Meisam Soleimani Malekan**.

Figure : Meisam Soleimani Malekan

Topological space, Hausdorffness, compactness

- By a topological space we mean a set X with a subset $\mathcal{T}$ of the power set of X such that $\varnothing, X \in \mathcal{T}$ and $\mathcal{T}$ is closed under finite intersection and arbitrary union of its elements. The members of $\mathcal{T}$ are called open sets in X.

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- A topological space X is called compact if $X = \bigcup_{A \in \mathcal{O}} A$ for some family $\mathcal{O}$ of open subsets of X, then $X = \bigcup_{A \in \mathcal{F}} A$ for some finite subset $\mathcal{F}$ of $\mathcal{O}$.

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- By a topological group, we mean a group G endowed with a topology, where the map from $G \times G$ to G defined by $(x, y) \mapsto xy^{-1}$ is continuous, where $G \times G$ is endowed by the product topology obtained from the topology of G.

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Borel Algebra

Let X be a topological space. We denote by Σ_X the σ-algebra generated by open subsets of X and is called the Borel algebra of X, i.e., the smallest set containing X itself which is closed under complement and under countable unions. Elements of Σ_X are called Borel set.

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- (Countably Additive) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$, where $A_n \in \Sigma_G$ for all $n \in \mathbb{N}$ and $A_n \cap A_m = \varnothing$ whenever $n \neq m$.

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- (Outer Regular) The measure μ is outer regular on Borel sets $S \in \Sigma_G$, i.e., $\mu(S) = \inf\{\mu(U) \ : \ S \subseteq U, U \text{ open}\}$.
- (Inner Regular) The measure μ is inner regular on open sets $U \subseteq G$, i.e., $\mu(U) = \sup\{\mu(K) \ : \ K \subseteq U, K \text{ compact}\}$.

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- The unique normalized Haar measure of $G^{[k]} := G \times \cdots \times G$ (k times) is equal to $\mathbf{m}_G \times \cdots \times \mathbf{m}_G$ on the sets of the form $U_1 \times \cdots \times U_k$ of $G^{[k]}$, where $U_i \in \Sigma_G$, i.e. $\mathbf{m}_{G^{[k]}}(U_1 \times \cdots \times U_k) = \mathbf{m}_G(U_1) \cdots \mathbf{m}_G(U_k)$

Sets with positive Haar measures

Open Sets has positive Haar Measure

If G is a compact group, then $\mu(U) > 0$ for any non-empty open set of G; for, since $\varnothing \neq U$, it easily follows that $G = \bigcup_{g \in G} Ug$. Since G is compact, $G = \bigcup_{g \in T} Ug$ for some finite subset T of G. Now as μ is countably additive, $1 = \mu(G) \leq \sum_{g \in T} \mu(Ug)$ and since μ is invariant, $1 \leq |T|\mu(U)$. Hence $\mu(U) > 0$.

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Any Borel Set containing a non-empty Open Set has Positive Haar Measure

Let S be a Borel set such that $U \subseteq S$ for some open set U, then $\mu(U) \leq \mu(S)$ and so $\mu(S) > 0$.

Sets satisfying a nilpotent law with positive Haar measure

- For any group G and positive integer k, denote by $\mathcal{N}_k(G)$ the set
$$\{(x_1, \ldots, x_{k+1}) \in G^{k+1} \mid [x_1, \ldots, x_{k+1}] = 1\},$$
where $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and inductively $[x_1, \ldots, x_{n+1}] := [[x_1, \ldots, x_k], x_{n+1}]$ for all $n \geq 2$.

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- It is asked in
A. Martino, M. C. H. Tointon, M. Valiunas and E. Ventura, Probabilistic Nilpotence in infinite groups, to appear in Israel J. Math.
that:
(Question) Let G be a compact group, and suppose that $\mathcal{N}_k(G)$ has positive Haar measure in $G^{[k+1]}$. Does G have an open k-step nilpotent subgroup?

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- A Hausdorff compact totally disconnected topological group is called a profinite group.
- Question 1 has positive answer whenever $k = 1$ and G assume to be profinite. It follows from Theorem 1 of L. Lévai and L. Pyber, Profinite groups with many commuting pairs or involutions, Arch. Math. (Basel) 75 no. 7 (2000) 1-7.

Compact groups with $\mathbf{m}(\mathcal{N}_1) > 0$

- Question 1 has positive answer whenever $k = 1$. It follows from Theorem 1.2 of
  K. H. Hofmann and F. G. Russo, The probability that x and y commute in a compact group, Math. Proc. Cambridge Philos. Soc. 153 (2012), no. 3, 557-571.

Finitely generated profinite groups with $\mathbf{m}(\mathcal{N}_k) > 0$

- A topological group is called topologically finitely generated if it contains a dense (algebraically) finitely generated subgroup.

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- A topological group is called topologically finitely generated if it contains a dense (algebraically) finitely generated subgroup.
- The motivation of Question 1 is the following. It follows from A. Shalev, Probabilistically nilpotent groups. Proc. Amer. Math. Soc. 146 (2018), no. 4, 1529-1536.
  that if G is a topologically finitely generated profinite group such that $\mathbf{m}_{G^{[k+1]}}(\mathcal{N}_k(G)) > 0$ then G has an open k-step nilpotent subgroup.

Profinite groups with $\mathbf{m}(\mathcal{N}_k) > 0$

- The proposers of Question 1 answer positively it for profinite groups.

Compact groups with $\mathbf{m}(\mathcal{N}_2) > 0$

- We prove that Question 1 has positive answer for $k = 2$.

Compact groups with $\mathbf{m}(\mathcal{N}_2) > 0$

We first prove that

Right Lemma

Suppose that G is a group and $x_1, x_2, x_3, g_1, g_2, g_3 \in G$ are such that

$$\begin{aligned} 1 = [x_1, x_2, x_3] &= [x_1g_1, x_2g_2, x_3g_3] = [x_1g_1, x_2g_2, x_3] \\ &= [x_1g_1, x_2, x_3g_2] = [x_1g_1, x_2, x_3] = [x_1g_1, x_2, x_3g_3] \\ &= [x_1, x_2, x_3g_1] = [x_1, x_2g_2, x_3g_1] = [x_1, x_2g_2, x_3] \\ &= [x_1, x_2, x_3g_2] = [x_1, x_2g_2, x_3g_3] = [x_1, x_2, x_3g_3]. \end{aligned}$$

Then $[g_1, g_2, g_3] = 1$.

We need the “right version” of Theorem 2.3 of
M. Soleimani Malekan, A. A. and M. Ebrahimi, Compact groups with many elements of bounded order, J. Group Theory, 23 (2020) no. 6 991-998.
as follows.

Right Version

If A is a measurable subset with positive Haar measure in a compact group G, then for any positive integer k there exists an open subset U of G containing 1 such that $\mathbf{m}_G(A \cap Au_1 \cap \cdots \cap Au_k) > 0$ for all $u_1, \ldots, u_k \in U$.

Compact groups with $\mathbf{m}(\mathcal{N}_2) > 0$

Let $X := \mathcal{N}_2(G)$. By Theorem Right Version that there exists an open subset $U = U^{-1}$ of G containing 1 such that

$$X \cap X\bar{u}_1 \cap \cdots \cap X\bar{u}_{11} \neq \varnothing \quad (*)$$

for all $\bar{u}_1, \ldots, \bar{u}_{11} \in U \times U \times U$. Now take arbitrary elements $g_1, g_2, g_3 \in U$ and consider

$$\begin{aligned}
\bar{u}_1 &= (g_1^{-1}, g_2^{-1}, g_3^{-1}), \bar{u}_2 = (g_1^{-1}, g_2^{-1}, 1) \\
\bar{u}_3 &= (g_1^{-1}, 1, g_2^{-1}), \bar{u}_4 = (g_1^{-1}, 1, 1) \\
\bar{u}_5 &= (g_1^{-1}, 1, g_3^{-1}), \bar{u}_6 = (1, 1, g_1^{-1}) \\
\bar{u}_7 &= (1, g_2^{-1}, g_1^{-1}), \bar{u}_8 = (1, g_2^{-1}, 1) \\
\bar{u}_9 &= (1, 1, g_2^{-1}), \bar{u}_{10} = (1, g_2^{-1}, g_3^{-1}) \\
\bar{u}_{11} &= (1, 1, g_3^{-1}).
\end{aligned}$$

Compact groups with $\mathbf{m}(\mathcal{N}_2) > 0$

By $(*)$, there exists $(x_1, x_2, x_3) \in X$ such that all the following 3-tuples are in X.

$$(x_1g_1, x_2g_2, x_3g_3), (x_1g_1, x_2g_2, x_3), (x_1g_1, x_2, x_3g_2)$$
$$(x_1g_1, x_2, x_3), (x_1g_1, x_2, x_3g_3), (x_1, x_2, x_3g_1), (x_1, x_2g_2, x_3g_1)$$
$$(x_1, x_2g_2, x_3), (x_1, x_2, x_3g_2), (x_1, x_2g_2, x_3g_3), (x_1, x_2, x_3g_3).$$

Now Lemma implies that $[g_1, g_2, g_3] = 1$. Therefore the subgroup $H := \langle U \rangle$ generated by U is 2-step nilpotent. Since $H = \bigcup_{n\in\mathbb{N}} U^n$, H is open in G. This completes the proof.

THANKS FOR YOUR ATTENTION