Combinatorial approach for Burnside groups of relatively small odd exponents

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(joint work with Katrin Tent and Eliyahu Rips)

Let F be the free group with m free generators, $m \geqslant 2$. That is, $F = \langle x_1, \dots, x_m \rangle$. Let $\{w_1, w_2, \dots, w_i, \dots\}$ be all elements of F. Then

$$B(m,n) = \langle x_1, \dots, x_m \mid w_1^n, w_2^n, \dots, w_i^n, \dots \rangle,$$

is called *the free Burnside group of rank m and exponent n*.

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General Burnside problem. If $B(m, n)$ is finite?

Burnside problem: history

- E. Golod and I. Shafarevich, 1964, negative (without assuming that all elements have uniformly bounded order)
- P. Novikov, S. Adian, 1968, negative for all odd $n \geqslant 4381$
- S. Adian, 1975, negative for all odd $n \geqslant 665$
- Yu. Olshansky, 1982, negative for all odd $n > 10^{10}$
- S. Adian, 2015, negative for all odd $n \geqslant 101$
- E. Rips (starting from 1982)
- S. Ivanov, 1994, negative for all even $n \geqslant 2^{48}$
- I. Lysënok, 1996, negative for all even $n \geqslant 8000$
- $B(m, 2)$, $B(m, 3)$ (Burnside, 1902), $B(m, 4)$ (Sanov, 1940), and $B(m, 6)$ (Marshall Hall Jr., 1958) are finite for all m

We show that $B(m, n)$ is infinite for all odd $n \geqslant 297$.

Our method is based on iterated small cancellation theory and on the Rips's idea of the canonical form, which we put in a combinatorial framework.

$$B(m,n) = \langle x_1, \ldots, x_m \mid w_1^n, w_2^n, \ldots, w_i^n, \ldots \rangle,$$
$$\mathcal{H} = \langle w_1^n, w_2^n, \ldots, w_i^n, \ldots \rangle$$

Our general goal is to choose a unique representative of a special form in every coset $\mathrm{F}/\mathcal{H}$ and show that there are infinite number of such elements. The latter will be easy because of a special form of these representatives.

General scheme of the proof

We split the factorisation in $B(m, n) = \mathrm{F}/\mathcal{H}$ into a countable number of steps.

We start from F and take $\mathcal{R}_1 \subseteq \{w_1^n, \ldots, w_i^n, \ldots\}$ and choose a canonical representative in every coset of $\mathrm{F}/\langle\mathcal{R}_1\rangle$, $\mathcal{C}_1$ is a set of canonical representatives of rank 1.

We choose $\mathcal{R}_2' \subseteq \{\mathrm{can}_1(w_1^n), \ldots, \mathrm{can}_1(w_i^n), \ldots\}$ and choose a canonical representative in every coset of $\mathcal{C}_1/\langle\mathcal{R}_2'\rangle$, $\mathcal{C}_2$ is a set of canonical representatives of rank 2, etc.

$$\mathrm{F} \longrightarrow \mathrm{F}/\langle\mathcal{R}_1\rangle \cong \mathcal{C}_1 \longrightarrow \mathcal{C}_1/\langle\mathcal{R}_2'\rangle \cong \mathcal{C}_2 \longrightarrow \mathcal{C}_2/\langle\mathcal{R}_3'\rangle \cong \mathcal{C}_3 \longrightarrow \ldots$$

Using the above sequence, we define $\mathrm{can}(A)$ for every $A \in \mathrm{F}$. If $A\mathcal{H} = B\mathcal{H}$, then $\mathrm{can}(A) = \mathrm{can}(B)$.

$$A \mapsto A\langle\mathcal{R}_1\rangle \mapsto \mathrm{can}_1(A) \mapsto \mathrm{can}_1(A)\langle\mathcal{R}_2'\rangle \mapsto \mathrm{can}_2(\mathrm{can}_1(A)) \mapsto \ldots$$

Let us consider first two steps $\mathrm{F} \longrightarrow \mathrm{F}/\langle \mathcal{R}_1 \rangle \cong \mathcal{C}_1 \longrightarrow \mathcal{C}_1/\langle \mathcal{R}'_2 \rangle \cong \mathcal{C}_2$.

The step $\mathcal{C}_1 \longrightarrow \mathcal{C}_1/\langle \mathcal{R}'_2 \rangle \cong \mathcal{C}_2$ is almost a general step of induction.

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$\mathcal{R}_1 = \{x^n \in \mathrm{F} \,| x^{\pm 1}$ *and its cyclic shifts do not contain* $a^{\tau_1}, a \in \mathrm{F}\}, \tau_1 = 7$.

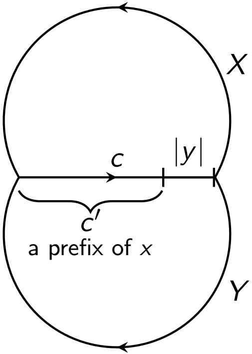
We are interested in common parts of relators from $\mathcal{R}_1$.

Lemma

Assume x^n, y^n *are two reduced words,* $|x| \geqslant |y|$, $x \neq y^k$, *and* $x^n = cX$, $y^n = cY$. *Then* $|c| < |x| + |y|$.

Corollary

Assume $x^n, y^n \in \mathcal{R}_1$, $|x| \geqslant |y|$, $x \neq y^k$, and $x^n = cX$, $y^n = cY$. Then $|c| < 2|x|$ and $|c| < (\tau_1 + 1)|y|$.

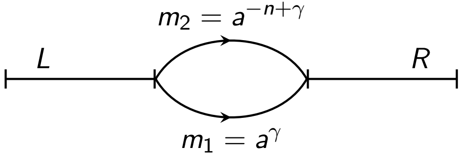


If $|c| \geqslant (\tau_1 + 1)|y|$, then c' contains y^{τ_1}. This contradicts to the assumption $x^n \in \mathcal{R}_1$.

Structure of $\mathrm{F}/\langle\mathcal{R}_1\rangle$

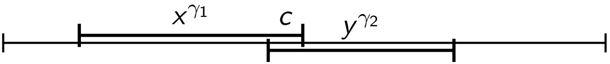
Let $U \in \mathrm{F}$ be a reduced word. We consider subwords of U of the form a^γ, where $a^n \in \mathcal{R}_1$, such that a^γ can not be prolonged to bigger fractional power.

Suppose $R = a^n \in \mathcal{R}_1$, $R = m_1 m_2^{-1}$. Let $U = Lm_1R$. The transition from $U = Lm_1R$ to Lm_2R representing the same element of $\mathrm{F}/\langle\mathcal{R}_1\rangle$. This transition is called *a turn*.

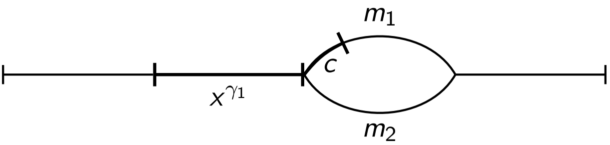


Two words represent the same element in $\mathrm{F}/\langle\mathcal{R}_1\rangle$ if and only if they are connected by a sequence of turns.

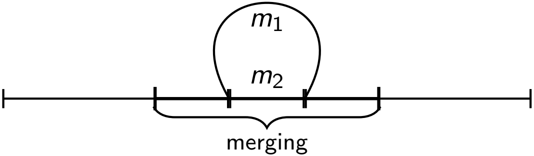
How turn influences another maximal occurrences



If $|x| \geqslant |y|$, then $|c| < (\tau_1 + 1)|y|$ and $|c| < 2|x|$.



$\Lambda(m_1) \geqslant \tau_1 + 1$, $\Lambda(m_2) \geqslant \tau_1 + 1$ (Λ-measure is a fractional number of periods of the corresponding relator)



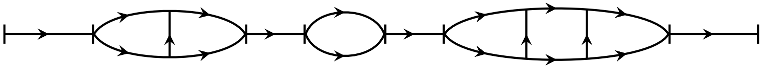
$\Lambda(m_2) < \tau_1 + 1$

Semicanonical words and one-layer maps

A word U is called λ-*semicanonical* is Λ-measure of every occurrence of a subword from $\mathcal{R}_1$ is $\leqslant \lambda$.

If $\lambda < n - (\tau_1 + 1)$, then there are no merging in a result of a turn in a λ-semicanonical word.

If we perform a sequence of turns in a λ-semicanonical word, we can see the results in the following picture

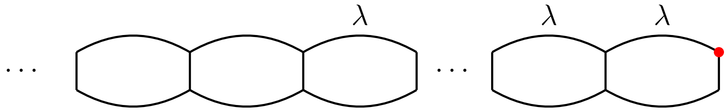


Further we consider such one-layer maps with outer sides $\leqslant \lambda$.

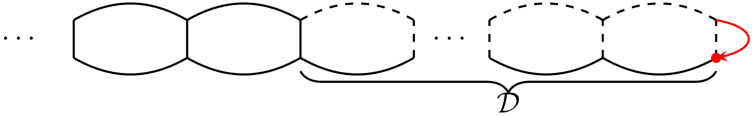
The canonical form in rank 1

We have to choose a canonical representative inside a one layer-map. Seems natural to take everywhere the smallest side. However, this is not good enough for our needs.

Let U be a λ-semicanonical word and consider its one-layer map.



Consider UX, that is, some small changes of U from the right.

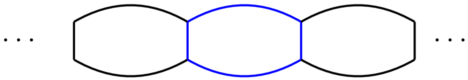


The canonical form in rank 1

We need a control over a structure of $\mathcal{D}$. We apply a special condition on this "domino part" of the one-layer map.

If the condition is satisfied, we take into account cells in $\mathcal{D}$ in U. If not, we erase them in U in advance.

After that we choose the smallest side using the rest cells.

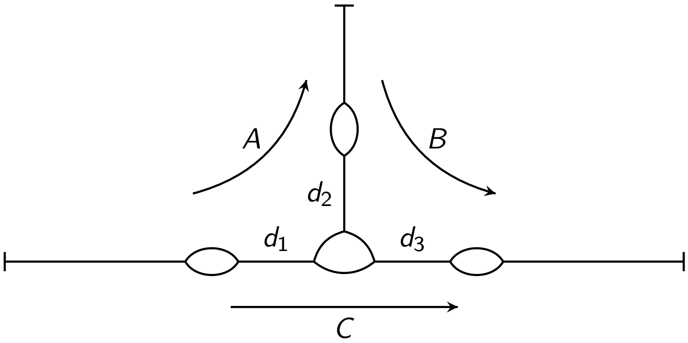


If $A \in \mathrm{F}$, then first we make it λ-semicanonical and then choose a canonical form in the corresponding one-layer map.

We show that if A and B represent the same element in $\mathrm{F}/\langle \mathcal{R}_1 \rangle$, then they belong to the same one-layer map. This yields that can_1 is well defined and $\mathrm{can}_1(A) = \mathrm{can}_1(B)$.

Multiplication of canonical words in rank 1

Let $C = \mathrm{can}_1(AB)$. Then we have the following picture (any of bubbles can be absent):



d_1, d_2, d_3 do not contain cubic powers formed by maximal occurrences of rank 1.

Structure of $\mathcal{C}_1/\langle\mathcal{R}'_2\rangle$

We make also a canonical form of a cyclic word. This is the same procedure but in a cyclic one-layer map. Let $\mathcal{C}_1^{\mathrm{c}}$ be canonical forms of cyclic words.

$$\mathcal{R}_2 = \Big\{x^n \in \mathcal{C}_1^{\mathrm{c}} \,|\, \textit{if } x^{\pm 1} \textit{or its cyclic shift contains } a^{\tau_1}, \\ \textit{then } a^n \in \mathcal{R}_1, a \in \mathrm{F}\Big\}.$$

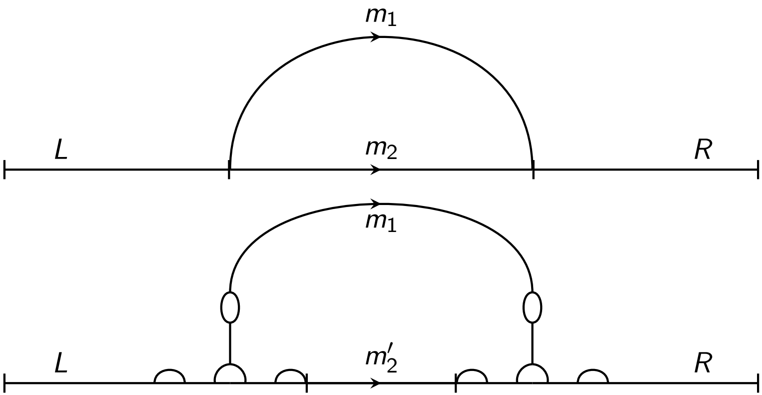
Moreover, one can show that if $x^n \in \mathcal{R}_2$, then there exists a cyclic shift of x that contains a^{τ_1}, $a^n \in \mathcal{R}_1$.

Lemma

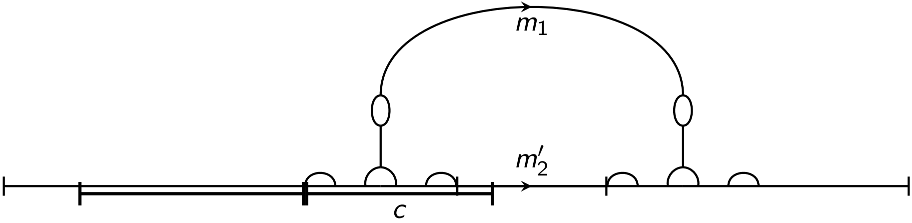
Assume $x^n, y^n \in \mathcal{R}_2$, $|x| \geqslant |y|$, $x \neq y^k$, and $x^n = cX$, $y^n = cY$. Then $|c| < 2|x|$ and $|c| < (\tau_1 + 1)|y|$.

Turn of rank 2

$U \in \mathcal{C}_1$, $U = Lm_1R$, $m_1m_2^{-1} \in \mathcal{R}_2$. We make a transformation $Lm_1R \to Lm_2R \to \mathrm{can}_1(Lm_2R)$.

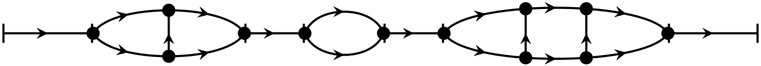


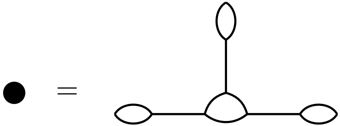
One can show that two words represent the same element of $\mathcal{C}_1/\langle\mathcal{R}'_2\rangle$ if and only they are connected by a sequence of turns of rank 2.



The key fact. $\Lambda(c) < 7 + \tau_1 + 1 = 7 + 7 + 1 = 15.$

So, we can use the same argument as in rank 1, but for a bit more complicated turns and one-layer-maps.

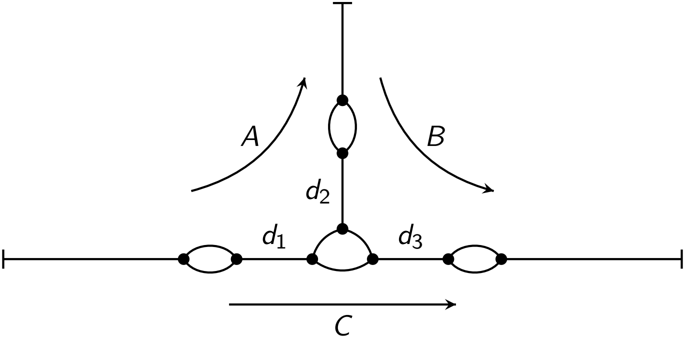


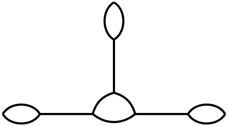


a canonical triangle of rank 1.

Multiplication of canonical words in rank 2

$C = \mathrm{can}_2(\mathrm{can}_1(AB))$.



● $=$  a canonical triangle of rank 1.

A global canonical form

Let $A \in \mathrm{F}$. Consider a sequence

$$A \mapsto \mathrm{can}_1(A) \mapsto \mathrm{can}_2(\mathrm{can}_1(A)) \mapsto \dots$$

It stabilizes after a finite number of steps. The resulting word is called *a global canonical form of A* and is denoted by $\mathrm{can}(A)$.

We show that A and B represent the same element in the group $B(m, n)$ if and only if $\mathrm{can}(A) = \mathrm{can}(B)$.

It is clear that if A is a cubic-free word, then $A = \mathrm{can}(A)$. Hence $|B(m, n)|$ is not less than number of cubic-free words. However, it is known that there are infinitely many different such words. Thus, $B(m, n)$ is infinite.

All these calculations work for odd $n \geqslant 297$.