New Criteria for solvability, nilpotency and other properties of finite groups

Marcel Herzog
School of Mathematical Sciences
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TEL AVIV UNIVERSITY
אוניברסיטת תל-אביב

ISRAEL

Introduction.

In this talk we shall describe some recent criteria for solvability, nilpotency and other properties of finite groups G, based either on the orders of the elements of G or on the orders of the subgroups of G. The various results will be described in the following Sections II, III and IV, each dedicated to one of the above mentioned properties of finite groups.

New criteria for solvability

We start with two criteria for solvability which were proved in the paper [1] of Patrizia Longobardi, Mercede Maj and myself. The first criterion was:

Theorem 1

Let G be a finite group of order n containing a subgroup A of prime power index p^s. Moreover, suppose that A contains a normal cyclic subgroup H and A/H is a cyclic group of order 2^r for some non-negative integer r. Then G is a solvable group.

Notice that since $[G : A] = p^s$, it follows that $G = AB$, where B is a p-group.

We continue with three remarks concerning this result. Here and later, G will denote a finite group.

Remark 1

Theorem 1 is a generalization of a special case of the following result of H. Wielandt and N. Ito (see Scott's book [2], Theorem 13.10.1):

Theorem 2

If $G = AB$, where B is a nilpotent subgroup of G and A is a subgroup of G containing a cyclic subgroup H of index $[A : H] \leq 2$, then G is solvable.

If B is a p-group, then this result corresponds to Theorem 1 with $1 \leq 2^r \leq 2$, while in our case r is not bounded.

Remark 2

For the proof of Theorem 1 we used the following Szep's conjecture, which was proved by Elsa Fisman and Zvi Arad in [3]:

Theorem 3

If $G = AB$, where A and B are subgroups of G with non-trivial centers, then G is not a non-abelian simple group.

Remark 3

The proof of Theorem 3 relies on the classification of finite simple groups. Therefore our proof of Theorem 1 also relies on that classification. On the other hand, the proof of Theorem 2 by Wielandt and Ito does not rely on the classification.

Before continuing, we need to introduce some notation, which will be used also in the other sections. First,

$$\psi(G) = \sum_{x \in G} o(x),$$

where $o(x)$ denotes the order of x. This notation was introduced by H. Amiri, S.M. Jafarian Amiri and I.M. Isaacs in their 2009 paper [4]. They proved the following theorem;

Theorem 4

If G is a non-cyclic group of order n and C_n denotes the cyclic group of order n, then $\psi(G) < \psi(C_n)$.

In [5], P. Longobardi, M. Maj and myself determined the exact upper bound for $\psi(G)$ for non-cyclic groups G. We proved the following theorem.

Theorem 5

If G is non-cyclic group of order n, then

$$\psi(G) \leq \frac{7}{11}\psi(C_n),$$

and equality holds for the groups $G = C_k \times C_2 \times C_2$, where k denotes an arbitrary odd integer.

Later we proved that equality holds only for the above mentioned groups.

Our second criterion for solvability in [1] was:

$$\text{if} \quad |G| = n \quad \text{and} \quad \psi(G) \geq \frac{1}{6.68}\psi(C_n),$$

then G is solvable. Since $\psi(A_5) = \frac{211}{1617}\psi(C_{60}) < \frac{1}{6.68}\psi(C_{60})$, we conjectured that if $\psi(G) > \frac{211}{1617}\psi(C_n)$, then G is solvable and this result is the best possible. And indeed, M. Baniasad Azad and B. Khosravi proved in [6] the following theorem:

Theorem 6

If $|G| = n$ and $\psi(G) > \frac{211}{1617}\psi(C_n)$, then G is solvable and this lower bound is the best possible.

And what can we say about a non-solvable group G of order n satisfying $\psi(G) = \frac{211}{1617}\psi(C_n)$? In this case, A. Bahri, B. Khosravi and Z. Akhlaghi proved in [7] the following theorem:

Theorem 7

Let G be a non-solvable group of order n. Then

$$\psi(G) = \frac{211}{1617}\psi(C_n)$$

if and only if $G = A_5 \times C_m$, where $(m, 30) = 1$.

The next criterion for solvability will be based on the orders of the subgroups of G.

We shall consider now the following function of G:

$$\sigma_1(G) = \frac{1}{|G|} \sum_{H \leq G} |H|,$$

which was introduced by T. De Medts and M. Tărnăuceanu in the paper [8].
Tărnăuceanu conjectured in [9] that

$$\textit{if} \quad \sigma_1(G) < \frac{117}{20}, \quad \text{then } G \text{ is solvable.}$$

Since $\sigma_1(A_5) = \frac{117}{20}$, this upper bound is the best possible.

In [10], P. Longobardi, M. Maj and myself proved Tărnăuceanu's conjecture . We proved the following theorem:

Theorem 8

If

$$\sigma_1(G) = \frac{1}{|G|} \sum_{H \leq G} |H| < \frac{117}{20},$$

then G is solvable and this upper bound is the best possible.

We pass now to the section dealing with criteria for nilpotency.

New criteria for nilpotency

In [11], M. Tărnăuceanu proved the following theorem:

Theorem 9

If

$$\sigma_1(G) = \frac{1}{|G|} \sum_{H \leq G} |H| < 2 + \frac{4}{|G|},$$

then G is a nilpotent group.

Inspired by this result, he asked whether there exists a constant $c > 2$ such that if $\sigma_1(G) < c$, then G is nilpotent. In the same paper Tărnăuceanu showed that such c does not exist. In fact, he constructed an infinite sequence of non-nilpotent groups (G_n), such that $\sigma_1(G_n)$ approaches 2 monotonically from above, as n tends to infinity.

Since a nilpotent group is a direct product of its Sylow subgroups, it follows that if G is nilpotent and $x, y \in G$ are of co-prime orders, then $o(xy) = o(x)o(y)$. But is the converse of this statement true? This is a natural question and it is very surprising that only in 2018 this fact was proved by Benjamin Baumslag and James Wiegold, using only elementary methods. They proved in [12] the following theorem.

Theorem 10

G is nilpotent if and only if

$$o(xy) = o(x)o(y)$$

for any $x, y \in G$ of co-prime orders.

Using the function $\psi(G) = \sum_{x \in G} o(x)$, Tărnăuceanu proved in [13] the following theorem.

Theorem 11

If $|G| = n$ *and*

$$\psi(G) > \frac{13}{21}\psi(C_n),$$

then G *is nilpotent. Moreover,* $\psi(G) = \frac{13}{21}\psi(C_n)$ *if and only if*

$$G = S_3 \times C_m \quad \text{with} \quad (m, 6) = 1.$$

It is interesting to notice that in the paper [14], dealing with non-cyclic groups of order $2m$, m odd, P. Longobardi, M. Maj and myself proved the following related result (see Theorem 7 in [14]):

Theorem 12

Let G be a non-cyclic group of order $n = 2m$, with m an odd integer. Then

$$\psi(G) \leq \frac{13}{21}\psi(C_n).$$

Moreover, $\psi(G) = \frac{13}{21}\psi(C_n)$ if and only if $m = 3m_1$ with $(m_1, 3) = 1$ and

$$G = S_3 \times C_{m_1}.$$

Since by Corollary 4 in [14] $\psi(G) < \frac{1}{2}\psi(C_n)$ if $n = |G|$ is odd, the above two theorems yield the following result: if G is a non-cyclic group of order n and $\psi(G) > \frac{13}{21}\psi(C_n)$, then G is nilpotent group with n divisible by 4. But Tărnăuceanu proved more. He proved the following theorem (see Corollary 1.2 in [13]):

Theorem 13

If $|G| = n$ *and*

$$\psi(G) > \frac{13}{21}\psi(C_n),$$

then

$$\frac{\psi(G)}{\psi(C_n)} \in \{\frac{27}{43}, \frac{7}{11}, 1\}$$

and one of the following statements holds, respectively:

1. $G = Q_8 \times C_m$, *where* m *is odd;*
2. $G = (C_2 \times C_2) \times C_m$, *where* m *is odd;*
3. G *is cyclic.*

Tărnăuceanu's Theorems 11 and 13 imply the following result:

Theorem 14

The four largest values of the fraction $\frac{\psi(G)}{\psi(C_{|G|})}$ on the class of finite groups G are:

$$1, \ \frac{7}{11}, \ \frac{27}{43}, \ \frac{13}{21}$$

in the decreasing order.

Another interesting criterion for nilpotency is the following theorem of Tărnăuceanu in [15]. He defined the function

$$\varphi(G) = |\{a \in G \mid o(a) = exp(G)\}|,$$

and he proved that

Theorem 15

G is nilpotent if and only if $\varphi(S) \neq 0$ for any section S of G.

Recall that a section of a group G is a homomorphic image of a subgroup of G. Moreover, if G is nilpotent, then $\varphi(G) > 0$.

The proof of Theorem 15 relies on the structure of minimal non-nilpotent groups. Tărnăuceanu also presented examples of **non-nilpotent groups** G which satisfy $\varphi(G) \neq 0$ and even $\varphi(H) \neq 0$ for all subgroups H of G. So considering only subgroups of G is not sufficient.

Another interesting characterization of nilpotency was
proved by Martino Garonzi and Massimiliano Patassini in their paper [16].
Let φ denote the Euler totient function. They proved the following
theorem

Theorem 16

Let $r < 0$ be a real number and let $|G| = n$. Then

$$\sum_{x \in G} \left(\frac{o(x)}{\varphi(o(x))} \right)^r \geq \sum_{x \in C_n} \left(\frac{o(x)}{\varphi(o(x))} \right)^r$$

and equality holds if and only if G is nilpotent. In particular, G is nilpotent if and only if

$$\sum_{x \in G} \left(\frac{\varphi(o(x))}{o(x)} \right) = \sum_{x \in C_n} \left(\frac{\varphi(o(x))}{o(x)} \right).$$

It is worthwhile to mention that if G is a nilpotent group of order n, then

$$\sum_{x\in G}\left(\frac{o(x)}{\varphi(o(x))}\right)^s=\sum_{x\in C_n}\left(\frac{o(x)}{\varphi(o(x))}\right)^s$$

for all real numbers s and by Theorem 16 the converse is true if $s<0$. It had been conjectured that the converse is true for all real numbers $s\neq 0$.

In [17], Tărnăuceanu considered the following function of G:

$$\psi''(G) = \frac{\psi(G)}{|G|^2}.$$

Among other results, some to be mentioned later, he proved the following theorem:

Theorem 17

If

$$\psi''(G) > \frac{13}{36} = \psi''(S_3),$$

then G is nilpotent.

Which groups satisfy $\psi''(G) = \frac{13}{36}$? This question will be considered in the next section.

Finally, we shall describe an amusing connection between group theory and number theory, discovered by Tom De Medts and Marius Tărnăuceanu . Let n denote a positive integer and let $\sigma(n)$ denote the sum of the divisors of n:

$$\sigma(n) = \sum_{d|n} d.$$

Recall that n is a deficient number if $\sigma(n) < 2n$ and a perfect number if $\sigma(n) = 2n$. Thus the set consisting of both the deficient numbers and the perfect numbers is characterized by the inequality

$$\sigma(n) \leq 2n.$$

Now let G be a group and denote by $C(G)$ the set of cyclic subgroups of G. Denote by S_1 and S_2 the following classes of groups:

$$S_1 = \{G \mid \sum_{H \leq G} |H| \leq 2|G|\}$$

and

$$S_2 = \{G \mid \sum_{H \in C(G)} |H| \leq 2|G|\}.$$

Clearly $S_1 \subseteq S_2$ and

$$\sum_{H \leq G} |H| = \sum_{H \in C(G)} |H|$$

if and only if G is a cyclic group.

De Medts and Tărnăuceanu proved in [8] the following theorem.

Theorem 18

Let G be a group of order n. Then the following statements hold.

1. $G \in S_1 = \{G \mid \sum_{H \leq G} |H| \leq 2|G|\}$ *if and only if G is cyclic and n is either a deficient or a perfect number.*
2. *G is a nilpotent group belonging to*

$$S_2 = \{G \mid \sum_{H \in C(G)} |H| \leq 2|G|\}$$

if and only if n is either a deficient or a perfect number.

Recent criteria for some other types of groups

In this section we shall list some criteria for a group to be cyclic, abelian and supersolvable.
In [17], Tărnăuceanu proved the following theorem. Recall that $\psi(G) = \sum_{x \in G} o(x)$ and $\psi''(G) = \frac{\psi(G)}{|G|^2}$.

Theorem 19

Let G be a finite group. Then the following statements hold.

1. *If $\psi''(G) > \frac{7}{16} = \psi''(C_2 \times C_2)$, then G is cyclic.*
2. *If $\psi''(G) > \frac{27}{64} = \psi''(Q_8)$, then G is abelian.*
3. *If $\psi''(G) > \frac{13}{36} = \psi''(S_3)$, then G is nilpotent.*
4. *If $\psi''(G) > \frac{31}{144} = \psi''(A_4)$, then G is supersolvable.*
5. *If $\psi''(G) > \frac{211}{3600} = \psi''(A_5)$, then G is solvable.*

3 was mentioned in Section 3 dealing with nilpotent groups. In this paper Tărnăuceanu stated the following open problem: determine all finite groups G for which $\psi''(G)$ takes on one of the values which appear in Theorem 19. He mentions that given a rational number $c \in (0,1)$, the main difficulty in finding groups G satisfying $\psi''(G) = c$ is to solve this problem for cyclic groups G.
A partial solution to this problem was supplied by M. Baniasad Azad, B. Khosravi and M. Jafarpour in their paper [18]. They proved the following theorem.

Theorem 20

Let G be a finite group. Then the following statements hold.

1. *If G is non-cyclic and $\psi''(G) = \frac{7}{16}$, then $G = C_2 \times C_2$.*
2. *If G is non-cyclic and $\psi''(G) = \frac{27}{64}$, then $G = Q_8$.*
3. *If G is non-cyclic and $\psi''(G) = \frac{13}{36}$, then $G = S_3$.*
4. *If G is non-supersolvable and $\psi''(G) = \frac{31}{144}$, then $G = A_4$.*
5. *If G is non-solvable and $\psi''(G) = \frac{211}{3600}$, then $G = A_5$.*

We shall conclude our treatment of the function $\psi''(G) = \frac{\psi(G)}{|G|^2}$, with the following two recent results.

The first result is the following theorem, which also appeared in the paper [18] of M. Baniasad Azad, B. Khosravi and M. Jafarpour.

Theorem 21

Let $p > 5$ be a prime and suppose that G is not a p-nilpotent group. Then

$$\psi''(G) \leq \frac{p^2+p+1}{4p^2} = \psi''(D_{2p}),$$

and equality holds if and only if $G = D_{2p}$.

The second result is the following theorem of M. Lazorec and M. Tărnăuceanu which appeared in their paper [19].

Theorem 22

The set $Im\psi" = \{\psi"(G) \mid G \text{ is a finite group}\}$ *is dense in* $[0,1]$.

The final topic of this talk are the cyclic subgroups of G. First we shall define three relevant functions.

Definitions

Let G be a finite group.

1. $C(G)$ denotes the set of cyclic subgroups of G.
2. $\alpha(G) = \frac{|C(G)|}{|G|}$.
3. $o(G)$ denotes the average order in G. Hence

$$o(G) = \frac{1}{|G|} \sum_{x \in G} o(x) = \frac{\psi(G)}{|G|}.$$

First we notice that in the multiset $\{\langle x\rangle \mid x \in G\}$, each $\langle x\rangle$ appears exactly $\varphi(o(x))$ times, where φ denotes the Euler totient function. Consequently, each $x \in G$ contributes $\frac{1}{\varphi(o(x))}$ to $|C(G)|$ and therefore

$$|C(G)| = \sum_{x \in G} \frac{1}{\varphi(o(x))}.$$

Hence $|C(G)|$, $\alpha(G) = \frac{|C(G)|}{|G|}$ and $o(G) = \frac{1}{|G|}\sum_{x\in G} o(x)$ are completely determined by the orders of the elements of the group G.
In his paper [20], Andrei Jaikin-Zapirain proved the following result concerning $o(G) = \frac{1}{|G|}\sum_{x\in G} o(x)$ (see his Lemma 2.7 and Corollary 2.10).

Theorem 23

If G is a finite group, then

$$k(G) \geq o(G) \geq o(Z(G)),$$

where $k(G)$ denotes the number of the conjugacy classes of G.

In his paper [21], Tărnăuceanu proved that the function $\alpha(G) = \frac{|C(G)|}{|G|}$ satisfies the reversed inequality:

Theorem 24

If G is a finite group, then

$$\alpha(G) \leq \alpha(Z(G)).$$

He also determined which groups satisfy the equality. In particular, he showed that such groups are 4-abelian, namely. $(xy)^4 = x^4y^4$ hoilds for all $x, y \in G$.

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The lecture is now complete.

THANK YOU for your ATTENTION!