Finite Coverings of Semigroups

Luise-Charlotte Kappe
Binghamton University
Joint with Casey Donoven

Ischia Group Theory Meeting 2020

Exercise

No group is the union of two proper subgroups.

Exercise

No group is the union of two proper subgroups.

Theorem

A group G is the union of three proper subgroups if and only if G has a homomorphic image isomorphic to the Klein 4-group.

Exercise

No group is the union of two proper subgroups.

Theorem

A group G is the union of three proper subgroups if and only if G has a homomorphic image isomorphic to the Klein 4-group.

G. Scorza, I gruppi che possone pensarsi come somma di tre lori sottogruppi, Boll. Un. Mat. Ital., 5 (1926), 216-218.

Exercise

No quasigroup is the union of two proper subquasigroups.

Exercise

No quasigroup is the union of two proper subquasigroups.

Loop = group without associativity

Exercise

No quasigroup is the union of two proper subquasigroups.

Loop = group without associativity
Quasigroup = loop without identity

Exercise

No quasigroup is the union of two proper subquasigroups.

Loop = group without associativity
Quasigroup = loop without identity

Definition

An algebraic structure A has a finite covering by proper algebraic substructres of A if A is the union of finitely many proper substructures. The covering number of an algebraic structure A, denoted $\sigma(A)$, is the minimum number of proper substructures whose union is A.

Theorem

For every integer $n > 2$, there exists a loop L with $\sigma(L) = n$

Theorem

For every integer $n > 2$, there exists a loop L with $\sigma(L) = n$

S. M. Gagola III and L. C. Kappe, On the covering number of loops, Expositiones Mathematicae, 34 (2016) 436-447

Theorem

A group is the union of finitely many proper subgroups if and only if it has a finite non-cyclic homomorphic image.

Theorem

A group is the union of finitely many proper subgroups if and only if it has a finite non-cyclic homomorphic image.

B.H. Neumann, Groups covered by finitely many cosets, Publ. Math. Debrecen, 3 (1954), 227-242

Theorem

The integers $n \leq 26$ which are not covering numbers of a group are 2, 7, 11, 19, 21, 22, and 25.

Theorem

The integers $n \leq 26$ which are not covering numbers of a group are 2, 7, 11, 19, 21, 22, and 25.

Tomkinson, Garonzi et. al.

Theorem

The integers $n \leq 26$ which are not covering numbers of a group are 2, 7, 11, 19, 21, 22, and 25.

Tomkinson, Garonzi et. al.

Theorem

The integers between 26 and 129 which are not covering numbers of a group are 27, 34, 35, 37, 39, 41, 43, 45, 47, 49, 51, 52, 53, 55, 56, 58, 59, 61, 66, 69, 70, 75, 76, 77, 78, 79, 81, 83, 87, 88, 89, 91, 93, 94, 95, 96, 97, 99, 100, 101, 103, 105, 106, 107, 109, 111, 112, 113, 115, 116, 117, 118, 119, 120, 123, 124, and 125.

Theorem

The integers $n \leq 26$ which are not covering numbers of a group are 2, 7, 11, 19, 21, 22, and 25.

Tomkinson, Garonzi et. al.

Theorem

The integers between 26 and 129 which are not covering numbers of a group are 27, 34, 35, 37, 39, 41, 43, 45, 47, 49, 51, 52, 53, 55, 56, 58, 59, 61, 66, 69, 70, 75, 76, 77, 78, 79, 81, 83, 87, 88, 89, 91, 93, 94, 95, 96, 97, 99, 100, 101, 103, 105, 106, 107, 109, 111, 112, 113, 115, 116, 117, 118, 119, 120, 123, 124, and 125.

M. Garonzi, L. C. Kappe, and E. Swartz, On Integers that are Covering Numbers of Groups, Experimental Mathematics, to appear

Exercise

There is no ring with covering number 2.

Exercise

There is no ring with covering number 2.

Theorem

A ring has finite covering number if and only if there exists a finite quotient with finite covering.

Exercise

There is no ring with covering number 2.

Theorem

A ring has finite covering number if and only if there exists a finite quotient with finite covering.

Theorem

Let R be a ring. If S is a subring of finite index, then S also contains a two-sided ideal of R which is also of finite index.

Exercise

There is no ring with covering number 2.

Theorem

A ring has finite covering number if and only if there exists a finite quotient with finite covering.

Theorem

Let R be a ring. If S is a subring of finite index, then S also contains a two-sided ideal of R which is also of finite index.

J. Lewin, Subrings of finite index in finitely generated rings, J. Algebra, 5 (1967), 84-88

Theorem

For every integer n with $2 < n < 13$, there exists a ring R with $\sigma(R) = n$.

Theorem

For every integer n with $2 < n < 13$, there exists a ring R with $\sigma(R) = n$.

Conjecture

There exists no ring with covering number 13.

Example

Let $\mathbb{N}$ be the semigroup of natural numbers under multiplication. Then $\sigma(\mathbb{N}) = 2$.

Definition

A semigroup is a set with an associative binary operation.

Definition

A semigroup is a set with an associative binary operation.

- Groups, functions with composition, numbers with addition/multiplication, words with concatenation...

Definition

A semigroup is a set with an associative binary operation.

- Groups, functions with composition, numbers with addition/multiplication, words with concatenation...

Definition

1. The covering number of a semigroup S with respect to semigroups, $\sigma_s(S)$, is the minimum number of proper subsemigroups of S whose union is S.

Definition

A semigroup is a set with an associative binary operation.

- Groups, functions with composition, numbers with addition/multiplication, words with concatenation...

Definition

1. The covering number of a semigroup S with respect to semigroups, $\sigma_s(S)$, is the minimum number of proper subsemigroups of S whose union is S.
2. The covering number of a semigroup S with respect to groups, $\sigma_g(S)$, is the minimum number of proper subgroups of S whose union is S.

Theorem (Donoven, K)

Let S be a finite semigroup.

- If S is monogenic, then $\sigma_s(S) = \infty$.
- If S is a group, then $\sigma_s(S) = \sigma_g(S)$.
- Otherwise, $\sigma_s(S) = 2$.

Definition

Let S be a semigroup and $x, y \in S$. Then $x \mathcal{J} y$ if only if $S^1 x S^1 = S^1 y S^1$.

Definition

Let S be a semigroup and $x, y \in S$. Then $x\mathcal{J}y$ if only if $S^1xS^1 = S^1yS^1$.

$\mathcal{J}$ is an equivalence relation known as a Green's Relation.

Definition

Let S be a semigroup and $x, y \in S$. Then $x\mathcal{J}y$ if only if $S^1xS^1 = S^1yS^1$.

$\mathcal{J}$ is an equivalence relation known as a Green's Relation.

There is a natural partial order $\leq_{\mathcal{J}}$ on the equivalence classes of $\mathcal{J}$: for $x, y \in S$, we have $J_x \leq_{\mathcal{J}} J_y$ if and only if $S^1xS^1 \subseteq S^1yS^1$.

Lemma

Let S be a semigroup and J be a maximal $\mathcal{J}$-class of S on the partial order. Then the set difference $S - J$ is a semigroup, provided $S - J \neq \emptyset$.

Lemma

Let S be a semigroup and J be a maximal $\mathcal{J}$-class of S on the partial order. Then the set difference $S - J$ is a semigroup, provided $S - J \neq \emptyset$.

Corollary

Let S be a semigroup with a maximal $\mathcal{J}$-class J such that $\langle J \rangle \neq S$, $\sigma_s(2)$.

Lemma

Let S be a semigroup and J be a maximal $\mathcal{J}$-class of S on the partial order. Then the set difference $S - J$ is a semigroup, provided $S - J \neq \emptyset$.

Corollary

Let S be a semigroup with a maximal $\mathcal{J}$-class J such that $\langle J \rangle \neq S$, $\sigma_s(2)$.

If $\langle J \rangle = S$, then either:

- $J = S$ and S is a Rees matrix semigroup, or
- $J \neq S$ and S surjects onto a Rees 0-matrix semigroup.

Definition

Let K and Λ be nonempty sets, G be a group, and P be a $|\Lambda| \times |K|$ matrix with entries in G. Then the Rees matrix semigroup $S = \mathcal{M}[K, G, \Lambda; P]$ is the set of triples $K \times G \times \Lambda$ with multiplication

$$(\kappa, g, \lambda)(\mu, h, \nu) = (\kappa, g p_{\lambda,\mu} h, \nu).$$

Definition

Let K and Λ be nonempty sets, G be a group, and P be a $|\Lambda| \times |K|$ matrix with entries in G. Then the Rees matrix semigroup $S = \mathcal{M}[K, G, \Lambda; P]$ is the set of triples $K \times G \times \Lambda$ with multiplication

$$(\kappa, g, \lambda)(\mu, h, \nu) = (\kappa, gp_{\lambda,\mu}h, \nu).$$

Proposition

Let $S = \mathcal{M}[K, G, \Lambda; P]$ be a Rees matrix semigroup. If $|K| > 1$ or $|\Lambda| > 1$, then $\sigma_s(S) = 2$. If $|K| = |\Lambda| = 1$, then S is a group.

Lemma

If G is a torsion group, then $\sigma_s(G) = \sigma_g(G)$.

Lemma

If G is a torsion group, then $\sigma_s(G) = \sigma_g(G)$.

Let G be a torsion group (e.g. finite) with subsemigroup T. Then for each $x \in T$, we have

- $id = x^{|x|} \in T$,
- $x^{-1} = x^{|x|-1} \in T$.

Thus, T is a group and $\sigma_s(G) = \sigma_g(G)$.

Lemma

If G is a torsion group, then $\sigma_s(G) = \sigma_g(G)$.

Let G be a torsion group (e.g. finite) with subsemigroup T. Then for each $x \in T$, we have

- $id = x^{|x|} \in T$,
- $x^{-1} = x^{|x|-1} \in T$.

Thus, T is a group and $\sigma_s(G) = \sigma_g(G)$.

Example

Let $C_\infty = \mathbb{Z}$, the integers under addition. Then $\sigma_g(\mathbb{Z}) = \infty$ but $\sigma_s(\mathbb{Z}) = 2$, since $\mathbb{Z} = \mathbb{Z}^- \cup (\mathbb{Z}^+ \cup \{0\})$.

Definition

Let K and Λ be nonempty sets, G be a group, and P be a $|\Lambda| \times |K|$ matrix with entries in $G \cup \{0\}$. Then the Rees 0-matrix semigroup, $S = \mathcal{M}^0[K, G, \Lambda; P]$, is the set $(K \times G \times \Lambda) \cup \{0\}$ with multiplication

$$(\kappa, g, \lambda)(\mu, h, \nu) = (\kappa, g p_{\lambda,\mu} h, \nu)$$

when $p_{\lambda,\mu} \neq 0$, and all other products are 0.

Definition

Let K and Λ be nonempty sets, G be a group, and P be a $|\Lambda| \times |K|$ matrix with entries in $G \cup \{0\}$. Then the Rees 0-matrix semigroup, $S = \mathcal{M}^0[K, G, \Lambda; P]$, is the set $(K \times G \times \Lambda) \cup \{0\}$ with multiplication

$$(\kappa, g, \lambda)(\mu, h, \nu) = (\kappa, g p_{\lambda,\mu} h, \nu)$$

when $p_{\lambda,\mu} \neq 0$, and all other products are 0.

Proposition

Let $S = \mathcal{M}^0[K, G, \Lambda; P]$ be a (regular) Rees 0-matrix semigroup. If $|K| > 1$ or $|\Lambda| > 1$, then $\sigma_s(S) = 2$. If $|K| = |\Lambda| = 1$, then S is monogenic.

Monoid

A monoid is a set with an associative binary operation with an identity 1.

Monoid

A monoid is a set with an associative binary operation with an identity 1.

Submonoid

A submonoid of a monoid is a subsemigroup that contains 1.

Monoid

A monoid is a set with an associative binary operation with an identity 1.

Submonoid

A submonoid of a monoid is a subsemigroup that contains 1.

Covering Numbers

The covering number of a monoid M with respect to submonoids, $\sigma_m(M)$, is the minimum number of proper submonoids of M whose union is M.

Theorem (Donoven, K)

Let M be a monoid.

- If M is a group, then $\sigma_m(M) = \sigma_s(M)$.
- If $M - \{1\}$ is a semigroup, then $\sigma_m(M) = \sigma_s(M - \{1\})$.
- Otherwise, $\sigma_m(M) = 2$.

Outline of proof for monoids:

Outline of proof for monoids:

Let R_1 be the set of elements of M with a right inverse.

Outline of proof for monoids:

Let R_1 be the set of elements of M with a right inverse.

If $M = R_1$, then M is a group.

Outline of proof for monoids:

Let R_1 be the set of elements of M with a right inverse.

If $M = R_1$, then M is a group.

If $R_1 = \{1\}$ and $M - R_1$ is non-empty, then $M - \{1\}$ is a semigroup.

Outline of proof for monoids:

Let R_1 be the set of elements of M with a right inverse.

If $M = R_1$, then M is a group.

If $R_1 = \{1\}$ and $M - R_1$ is non-empty, then $M - \{1\}$ is a semigroup.

Otherwise R_1 and $(M - R_1) \cup \{1\}$ are proper submonoids.

Corollary

If M is a monoid and not a group, then $\sigma_s(M) = 2$.

Corollary

If M is a monoid and not a group, then $\sigma_s(M) = 2$.

Theorem (Donoven)

Let G be a group. Then $\sigma_s(G) = 2$ if and only if G has a non-trivial left-orderable quotient.

Corollary

If M is a monoid and not a group, then $\sigma_s(M) = 2$.

Theorem (Donoven)

Let G be a group. Then $\sigma_s(G) = 2$ if and only if G has a non-trivial left-orderable quotient.

Open Question

For all groups G, is it true that $\sigma_s(G) = 2$ or $\sigma_s(G) = \sigma_g(G)$?

Inverse Semigroups

An inverse semigroup I is a set with an associative binary operation where for all $a \in I$ there is a unique $b \in I$ such that $aba = a$ and $bab = b$.

Inverse Semigroups

An inverse semigroup I is a set with an associative binary operation where for all $a \in I$ there is a unique $b \in I$ such that $aba = a$ and $bab = b$.

Covering Numbers

The covering number of an inverse subsemigroup I with respect to inverse subsemigroups, $\sigma_i(I)$, is the minimum number of proper inverse subsemigroups of I whose union is I.

Theorem (Donoven, K)

Let I be a finite inverse semigroup.

- If I is not generated by a single $\mathcal{J}$ class, then $\sigma_i(I) = 2$.
- If I is a group, then $\sigma_i(I) = \sigma_g(I)$.
- Otherwise, I surjects onto a Rees 0-Matrix semigroup $\mathcal{M}^0[K, G, K; P]$.
  - If $|K| = 2$ and $|G| = 1$, then $\sigma_i(I) = \infty$.
  - If $|K| = 2$ and $|G| \neq 1$, then $\sigma_i(I) = n + 1$ where n is the smallest index of a proper subgroup in G.
  - If $|K| \geq 3$, then $\sigma_i(I) = 3$.

Theorem (Donoven, K)

Let I be a finite inverse semigroup.

- If I is not generated by a single $\mathcal{J}$ class, then $\sigma_i(I) = 2$.
- If I is a group, then $\sigma_i(I) = \sigma_g(I)$.
- Otherwise, I surjects onto a Rees 0-Matrix semigroup $\mathcal{M}^0[K, G, K; P]$.
  - If $|K| = 2$ and $|G| = 1$, then $\sigma_i(I) = \infty$.
  - If $|K| = 2$ and $|G| \neq 1$, then $\sigma_i(I) = n + 1$ where n is the smallest index of a proper subgroup in G.
  - If $|K| \geq 3$, then $\sigma_i(I) = 3$.

Corollary

For all $n \geq 2$, there exists an inverse semigroup I such that $\sigma_i(I) = n$.

Thank you!