Compact groups with countable Engel sinks

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Engel groups

Notation: left-normed simple commutators

$$[a_1, a_2, a_3, \dots, a_r] = [...[[a_1, a_2], a_3], \dots, a_r].$$

Recall: a group G is an Engel group if for every $x, g \in G$,

$$[x,\ _n g] := [x, \underbrace{g, g, \dots, g}_{n}] = 1,$$

where g is repeated sufficiently many times depending on x and g.

Clearly, any locally nilpotent group is an Engel group.

The converse is not true in general,

but is a coveted result for particular classes of groups.

For example, finite Engel groups are nilpotent (Zorn's theorem).

Engel compact groups

J. Wilson and E. Zelmanov, 1992

Any Engel profinite group is locally nilpotent.

Proof relies on

Zelmanov's Theorem

If a Lie algebra L satisfies a nontrivial identity and is generated by d elements such that each commutator in these generators is ad-nilpotent, then L is nilpotent.

Yu. Medvedev, 2003

Any Engel compact (Hausdorff) group is locally nilpotent.

Generalizations using Engel sinks

Definition

An Engel sink of an element $g \in G$ is a set $\mathscr{E}(g)$ such that for every $x \in G$,

$$[x, \underbrace{g, g, \dots, g}_{n}] \in \mathscr{E}(g) \qquad \text{for all } n \geqslant n(x, g).$$

g is a (left) Engel element when $\mathscr{E}(g) = \{1\}$

Engel group when $\mathscr{E}(g) = \{1\}$ for all $g \in G$.

Note ambiguity of notation, as Engel sink may not be unique and a minimal Engel sink may not exist (unless there is a finite one).

Compact groups with finite Engel sinks

Reported at Ischia-2018:

Theorem 1 (EK & P. Shumyatsky, 2018)

Suppose that G is a compact (Hausdorff) group in which every element has a finite Engel sink. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

(G also has a locally nilpotent subgroup of finite index: $C_G(N)$)

Question (J. Wilson, Ischia 2018)

Is the same true for compact groups with countable Engel sinks?

The main result:

Theorem 2 (EK & P. Shumyatsky, 2020)

Suppose that G is a compact (Hausdorff) group in which every element has a countable Engel sink. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

(G also has a locally nilpotent subgroup of finite index: $C_G(N)$.)

Four steps in the proof

1. **Pronilpotent groups**, mainly pro-p groups.
2. **Prosoluble groups:** using properties of Engel sinks in coprime actions and a Hall–Higman–type theorem.
3. **Profinite groups:** bounding the non-prosoluble length and induction on this length.
4. **Compact groups:** reduction to profinite case using structure theorems for compact groups.

Theorem 3 (EK & P. Shumyatsky, 2020)

Suppose that G is a pronilpotent group in which every element has a countable Engel sink. Then G is locally nilpotent.

Main case: pro-p groups.

Lie ring methods are applied including Zelmanov's theorem on Lie algebras satisfying a polynomial identity and generated by elements all of whose products are ad-nilpotent.

Note: in Theorem 1 of 2018, the case of pronilpotent (pro-p) groups was easy: when every element of a pro-p group has a **finite** Engel sink, then it is easy to see that the group is Engel, so locally nilpotent by Wilson–Zelmanov.

But for countable Engel sinks, the case of pro-p groups requires substantial efforts.

Application of the Baire Category Theorem

Recall: G is a compact group;

$g \in G$ has a countable Engel sink $\mathscr{E}(g) = \{s_1, s_2, \dots\}$.

The sets $S_{ij} = \{x \in G \mid [x,\, {}_i g] = s_j\}$ are closed.

We have $G = \bigcup_{i,j} S_{ij}$ by the definition of the Engel sink.

By the Baire Category Theorem, one of these sets S_{ij} contains an open subset. In particular, in profinite case, we have

Lemma.

For every $g \in G$ there are $i \in \mathbb{N}$, $s \in \mathscr{E}(g)$, and a coset bN of an open normal subgroup N such that $[x,\, {}_i g] = s$ for all $x \in bN$.

Recall:

Lemma.

For every $g \in G$ *there are* $i \in \mathbb{N}$, $s \in \mathscr{E}(g)$, *and a coset* bN *of an open normal subgroup* N *such that* $[x,\, _i g] = s$ *for all* $x \in bN$.

Since G/N is finite, the coset Nb is invariant under conjugation by some g^{p^k}. Then

$$\begin{aligned} s^{g^{p^k}} &= [b,\, _i g]^{g^{p^k}} = [b^{g^{p^k}},\, _i g] \\ &= [nb,\, _i g] \quad \text{for some } n \in N \\ &= s. \end{aligned}$$

As a result, we have

Lemma.

For every $g \in G$ *there are* $i, k \in \mathbb{N}$ *and a coset* bN *of an open normal subgroup* N *such that* $[[x,\, _i g], g^{p^k}] = 1$ *for all* $x \in bN$.

Lie ring method for a pro-p group G

The Lie algebra $L_p(G) = \bigoplus_i G_i/G_{i+1}$ over $\mathbb{F}_p$

constructed with respect to the *Zassenhaus p-filtration*:

$$G_i = \langle g^{p^k} \mid g \in \gamma_j(G),\ jp^k \geqslant i \rangle \qquad \text{for} \quad i = 1, 2, \dots$$

Recall: Lemma.

For every $g \in G$ there are $i, k \in \mathbb{N}$ and a coset bN of an open normal subgroup N such that $[[x,\, _i g], g^{p^k}] = 1$ for all $x \in bN$.

From this lemma, for a **finitely generated** pro-p group G with countable Engel sinks, it is deduced that $L_p(G)$ satisfies the conditions of Zelmanov's theorem, so is nilpotent.

Hence G is p-adic analytic (Lazard 1965).

Plus satisfies a coset law $\Rightarrow$ soluble (Breuillard–Gelander 2007, 'topological Tits alternative').

Completion of the proof in the pronilpotent case

Finitely generated pro-p group G with countable Engel sinks.

Obtained that G is soluble.

Then induction on derived length $\Rightarrow$ G is nilpotent.

Then extended to pronilpotent G.

Coprime actions

Many steps of the proof for prosoluble groups, as well as for general profinite case, are based on facts about coprime actions.

For finite groups: if g is a **coprime** automorphism of a finite abelian group A, then

$$[A, g] = \{[a, g] \mid a \in A\} = \{[a, g, g] \mid a \in A\} = \cdots \subseteq \mathscr{E}(g).$$

Facts about coprime actions for finite groups have analogues for profinite groups.

One typical lemma in the proof for prosoluble groups:

Lemma

Let φ be a coprime automorphism of a pronilpotent group G. If all elements of $G\langle\varphi\rangle$ have countable Engel sinks, then $\gamma_\infty(G\langle\varphi\rangle)$ is finite and $\gamma_\infty(G\langle\varphi\rangle) = [G, \varphi]$.

Case of prosoluble groups

The proof is rather technical,

involves application of a ‘non-modular’ Hall–Higman–type theorem.

General profinite groups: non-prosoluble length

A profinite group G has finite *non-prosoluble length* at most l if G has a normal series

$$1 = L_0 \leqslant R_0 < L_1 \leqslant R_1 < \cdots \leqslant R_l = G$$

in which each quotient L_i/R_{i-1} is a (nontrivial) Cartesian product of non-abelian finite simple groups, and each quotient R_i/L_i is prosoluble (possibly trivial).

The same for finite: non-soluble length.

J. Wilson 1983

Let K be a normal subgroup of a finite group G. If a Sylow 2-subgroup Q of K has a coset tQ of exponent dividing 2^k, then the non-soluble length of K is at most k.

For profinite G with countable Engel sinks, such a coset law is provided by another application of the Baire Category Theorem using the lemma above.

Profinite groups: induction on non-prosoluble length

G profinite group with countable Engel sinks.

The above 'coset argument' is used to prove that G has finite non-prosoluble length.

Then induction on this length provides reduction to the prosoluble case.

Crucially, those Cartesian products of non-abelian finite simple groups can only be finite, as otherwise one can produce an element that cannot have countable Engel sink.

Compact groups

Recall **Structure theorems for compact groups:**

- The connected component G_0 of the identity is a divisible group (that is, for every $g \in G_0$ and every integer k there is $h \in G_0$ such that $h^k = g$).
- $G_0/Z(G_0)$ is a Cartesian product of simple compact Lie groups.
- G/G_0 is a profinite group.

Getting rid of simple Lie groups

Every simple Lie group has a section isomorphic to $SO_3(\mathbb{R})$,

in which it is easy to find an element

that cannot have a countable Engel sink.

Hence, G_0 is an abelian divisible group;

G/G_0 is profinite, for which Theorem 2 has already been proved.

Thus we have $G_0 < F < G$ with G_0 abelian divisible,
F/G_0 finite, and G/F locally nilpotent.

Next steps are similar to the proof of Theorem 1 of 2018,
... in the end use profinite case again...

Further results: compact groups with countable right Engel sinks

EK–Shumyatsky 2020

Further results: automorphisms of finite or profinite groups with restrictions on Engel sinks of their fixed points

C. Acciarri, EK, D. Silveira, P. Shumyatsky, et al.