AROUND CONTRANORMALITY

L.A. Kurdachenko

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Let G be a group. We consider the following subgroups, which have a significant impact on the structure of the group, but whose properties (especially in infinite groups) have not yet been studied deep enough.

If H, K are subgroups of a group G, then put $H^K = < H^x \mid x \in K >$. If $H \leq K$, then H^K is called *the normal closure of H in a subgroup K*. Note that H^K is the least normal subgroup of K including H. A subgroup H is normal in a group G if and only if $H = H^G$. In this sense, the subgroups H with the property $H^G = G$ are the complete opposites of the normal subgroups.

A subgroup H of a group G is called *contranormal* in G if $H = H^G$.

The term «a contranormal subgroup» has been introduced by J.S. Rose in his paper

RJ1968. J.S. Rose. *Nilpotent subgroups of finite soluble groups.* Math. Zeitschrift – 106(1968), 97 – 112.

Figuratively speaking, contranormal subgroups spread its influence over the whole group: all cyclic subgroups generated by the group elements are contained in their normal closures. In this regard, contranormal subgroups could be considered as antipodes to normal and subnormal subgroups.

Contranormal subgroups naturally appear in the following way. Let G be a group and H be a subgroup of G. Starting from the normal closure, we can construct the following canonical series.

Put $\nu_{0\,G}(H) = G$, $\nu_{1\,G}(H) = H^G$, and define $\nu_{\alpha+1\,G}(H) = H^{\nu_{\alpha\,G}(H)}$ for every ordinal α and $\nu_{\lambda\,G}(H) = \bigcap_{\beta<\lambda} \nu_{\beta\,G}(H)$ for all limit ordinals λ.

Thus we construct the *lower normal closure series*

$$G = \nu_{0\,G}(H) \geq \nu_{1\,G}(H) \geq \ \ldots\ \nu_{\alpha\,G}(H) \geq \nu_{\alpha+1\,G}(H) \geq \ldots\ \nu_{\gamma\,G}(H) = D$$

of a subgroup H in the group G. By the construction, $\nu_{\alpha+1\,G}(H)$ is a normal subgroup of $\nu_{\alpha\,G}(H)$ for all ordinals $\alpha < \gamma$. The last term D of this series has the property $H^D = D$. The last term $\nu_{\gamma\,G}(H)$ of this series is called the *lower normal closure of a subgroup H in G.*

We note that every subgroup H is contranormal in its lower normal closure in G, and lower normal closure of H in G is a descendant subgroup of G.

As we can see by the definition, contranormal subgroups are antipodes not only to normal subgroups, they are antipodes to subnormal and descendant subgroups: a contranormal subgroup H of a group G is subnormal (respectively descendant) if and only if $H = G$.

If G is a group and H is a contranormal subgroup of G, then every subgroup K including H is contranormal in G. In particular, if H and L are contranormal subgroups of G, then the subgroup $< H, L >$ is also contranormal in G. However, the intersection of two contranormal subgroups is not always contranormal. For example, in the group A_4 every Sylow 3 – subgroup is contranormal, but the intersection of every two Sylow 3 – subgroups of A_4 is trivial, so that it is not contranormal. Note also that if M is a maximal not normal subgroup of G, then M is a contranormal subgroup of G.

The important examples of contranormal subgroups are abnormal subgroups. Recall that a subgroup H of a group G is said to be *abnormal* in G if $g \in < H, H^g >$ for every element g of a group G.

Note that these two types of subgroups do not coincide, and the influence of these subgroups on the structure of a group is different.

We note that a finite group is nilpotent if and only if it does not include proper contranormal subgroups. Indeed, in such a group every maximal subgroup is normal. But in an infinite group the situation is different. If G is an infinite locally nilpotent group, then G does not include proper abnormal subgroups. However, a locally nilpotent group can include proper contranormal subgroups. Moreover, there are the examples of hypercentral groups with proper contranormal subgroups.

For example:

Let D be a divisible abelian 2 – subgroup. Then D has an automorphism ι such that $\iota(d) = d^{-1}$ for each element $d \in D$. Define a semidirect product $G = D \leftthreetimes \langle b \rangle$ such that $d^b = \iota(d) = d^{-1}$ for each element $d \in D$. It is not hard to see that the subgroup $\langle b \rangle$ is contranormal. We note that group G is not nilpotent, however G is hypercentral and abelian – by – finite. Besides, a contranormal subgroup $\langle b \rangle$ is ascendant.

Another example. Let D be a Prüfer 2 – subgroup and κ be a non – trivial automorphism of D. Then κ has an infinite order, a natural semidirect product $G = D \leftthreetimes \langle \kappa \rangle$ is a hypercentral group and subgroup $\langle \kappa \rangle$ is contranormal in G. As in the previous example, the subgroup $\langle \kappa \rangle$ is ascendant.

Therefore, the following natural question appears:

What can we say about a group which does not include proper contranormal subgroups?

We say that a group G is ***contranormal – free*** if G does not include proper contranormal subgroups. Note that in this situation we cannot speak of the nilpotency of the group, moreover, groups without contranormal subgroups can be very far from nilpotent.

Indeed, the groups whose subgroups are subnormal do not include proper contranormal subgroups. Such groups are locally nilpotent, but can be not nilpotent. In this connection, it is better to recall the example, which has been constructed by H. Heineken and A. Mohamed in the paper

HM 1968. Heineken H., Mohamed A. *A group with trivial centre satisfying the normalizer condition.* J. Algebra 10(3) (1968), 368 – 376.

This is a p – group H with the following properties:

H includes a normal elementary abelian p – subgroup A such that H/A is a Prüfer p – group; every proper subgroup of H is subnormal in G, $\zeta(H) = <1>$.

As other examples we consider groups, which are close to nilpotent.

A periodic group G is said to be *Sylow – nilpotent* if G is locally nilpotent and a Sylow p – subgroup of G is nilpotent for each prime p.

It is not hard to see that every Sylow – nilpotent group does not include proper contranormal subgroups.

Some types of contranormal – free groups have been studied in the following papers

KS2003. Kurdachenko L.A., Subbotin I.Ya. ***Pronormality, contranormality and generalized nilpotency in infinite groups.*** **Publicacions Matemàtiques, 2003, 47, number 2, 389 – 414**

KOS2009. Kurdachenko L.A., Otal J. and Subbotin I.Ya. ***Criteria of nilpotency and influence of contranormal subgroups on the structure of infinite groups.*** **C Turkish J. Math. – 33 (2009), 227 – 237.**

KOS2010. Kurdachenko L.A., Otal J. and Subbotin I.Ya. ***On influence of contranormal subgroups on the structure of infinite groups.*** **Communications in Algebra – 37 (2010), 4542 – 4557.**

WB2020. Wehrfritz B.A.F. ***Groups with no proper contranormal subgroups.*** **Publicacions Matemàtiques, 2020, 64, 183 – 194**

In particular, the following results have been obtained in [**KS2003**].

Let G be a Chernikov group. If G is contranormal – free, then G is nilpotent.

Let G be a locally finite group, every Sylow p – subgroup of which is Chernikov for every prime p. If G is contranormal – free, then G is Sylow – nilpotent.

Recall that a group G is called *hyperfinite*, if G has an ascending series of normal subgroups, whose factors are finite.

The following results have been obtained in the paper

Kurdachenko L.A., Longobardi P., Maj M. *On the structure of some locally nilpotent groups without contranormal subgroups – 2021*

Let G be a group, H be a locally nilpotent normal subgroups of G such that G/H is hyperfinite. If G is contranormal – free, then G is locally nilpotent.

As corollaries we obtain

Let G be a periodic group, H be a normal locally nilpotent subgroups of G such that G/H is nilpotent. If G is contranormal – free, then G is locally nilpotent.

Let G be a periodic group and H be a normal locally nilpotent subgroup such that G/H is a Chernikov group. If G is contranormal – free, then G is locally nilpotent.

Let G be a locally finite group and H be a normal locally nilpotent subgroup such that the Sylow p – subgroups of G/H are Chernikov for all prime p. If G is contranormal – free, then G is locally nilpotent.

Let G be a hyperfinite group. If G is contranormal – free, then G is hypercentral.

Let G be a periodic group, H be a normal nilpotent subgroups of G such that G/H is nilpotent and $\Pi(H) \cap \Pi(G/H) = \varnothing$. If G is contranormal – free, then G is nilpotent.

If G/H is a Chernikov group, then we obtain assertion (iii) of the paper [**WB2020**].

Let A be a torsion – free abelian normal subgroup of a group G. We say that A is *rationally irreducible with respect to G* or that A is *G – rationally irreducible* if, for every nontrivial G – invariant subgroup B of A, the factor – group A/B is periodic.

Let G be a group and A be a normal nilpotent subgroup of G such that G/A is a Chernikov π – group. Suppose that A has a finite series of G – invariant subgroups

$$A = A_0 \geq A_1 \geq \ldots \geq A_j \geq A_{j+1} \geq \ldots \geq A_t = \langle 1 \rangle$$

every factor A_j / A_{j+1} of which satisfies one of the following conditions:

A_j / A_{j+1} is torsion – free and G – rationally irreducible;
A_j / A_{j+1} is a periodic π′ – group;
A_j / A_{j+1} is a Chernikov π – group, $0 \leq j \leq t - 1$.

If G is contranormal – free, then G is nilpotent.

This result is a quite broad generalization of the main result the above mentioned paper of B.A.F. Wehrfritz [**WB2020**].

We show one more result of such a nature, which has been obtained in a paper [**KOS2010**].

Let G be a group and C be a normal subgroup of G. Then C is said to be *G - minimax* if C has a finite series of G – invariant subgroups whose infinite factors are abelian and either satisfy Min – G or Max – G.

Let G be a group and A be a normal G – minimax subgroup of G such that G/A is a nilpotent group of finite section rank. If G is contranormal – free, then G is nilpotent.

In the paper [**WB2020**], B.A.F. Wehrfritz proved in the following result

Let G be a nilpotent – by – finite group. If G is contranormal – free, then G is nilpotent.

The following results has been obtained recently in the paper

Dixon M.R., Kurdachenko L.A., Subbotin I.Ya. ***On the structure of some contranormal – free groups – 2021***

Let G be a group and H be a nilpotent normal subgroup of G such that G/H is finitely generated and soluble – by – finite. If G is contranormal – free, then G is hypercentral.

Let G be a locally generalized radical group, having finite section rank. If G is contranormal – free, then G is hypercentral group, having hypercentral length at most $\omega + k$ for some positive integer k (here ω is the first infinite ordinal). Moreover, every its factor – group G/H such that $\Pi(G/H)$ is finite, is nilpotent.

In particular, if the set $\Pi(G/H)$ is finite, then G is nilpotent. This assertion has been proved in the paper [**KOS2010**] and in the paper [**WB2020**].

As we have seen above, there exists a hypercentral abelian – by – finite group having proper contranormal (and even finite contranormal) subgroups. Therefore, the question about the structure of abelian – by – finite groups having proper contranormal subgroup is naturally appears. Here we can show the following result

Let G be a locally nilpotent group and A be a normal abelian subgroup of G such that the factor – group G/A is finite. If G includes a proper contranormal subgroup C, then $C = B K$ where $B \leq A$, K is a finitely generated subgroup such that $G = AK$ and $A = B[K, A]$. In particular, the factor – group A/B has a finite contranormal subgroup KB/B.

Thus, we naturally came to abelian – by – finite groups having finite contranormal subgroups. The following result gives a description of hypercentral groups, which includes a finite contranormal subgroup.

Let G be a hypercentral group. If G includes a finite contranormal subgroup, then G is periodic and satisfies the following conditions:

(i) $G = VC$ where V is a normal divisible abelian subgroup and C is a finite contranormal subgroup of G;

(ii) $\Pi(G) = \Pi(C)$, in particular a set $\Pi(G)$ is finite;

(ii) V has a family of G – invariant G – quasifinite subgroups $\{ D_\mu \mid \mu \in M \}$ such that

$V = < D_\mu \mid \mu \in M >$,

(iii) $[D_\mu, C] = D_\mu$ for all $\mu \in M$, in particular, $[V, C] = V$.

This result also has been obtained in the paper

Kurdachenko L.A., Longobardi P., Maj M. *On the structure of some locally nilpotent groups without contranormal subgroups – 2020*

And the following last result has been obtained in the paper

Kurdachenko L.A., Semko N.N. ***On the structure of some groups having finite contranormal subgroups – 2021***

Let G be a group and A be a normal abelian subgroup of G such that G/A is nilpotent. Suppose that G includes a finite contranormal p - subgroup C where p is a prime. Then G satisfies the following conditions:

(i) $G = D \leftthreetimes P$ *where D is a normal abelian subgroup and P is a p – subgroup;*
(ii) P is a hypercentral abelian – by - finite p – subgroup, having finite contranormal subgroup;
(iii) $D = D^p$*;*
(iv) Tor(D) = Q is a G - hypereccentric p′ – subgroup;
(v) every G – chief p′ – factor of D is G – eccentric;
(vi) If K, L are pure G – invariant subgroup of D such that $T \leq K \leq L$*, then* $L/K = [L/K, G]$*;*
(vii) D has an ascending series of G – invariant subgroups

$$T = D_0 \leq D_1 \leq . . . D_\alpha \leq D_{\alpha+1} \leq . . . D_\gamma = D$$

each factor $D_{\alpha+1}/D_\alpha$ *of which is torsion – free G – eccentric and G – rationally irreducible, for every* $\alpha < \gamma$*.*