Explicit Examples of Algebraically Closed Groups

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Algebraically Closed Groups

A group G is said to be **existentially closed (algebraically closed)** if every finite system

$$u_i(g_1, \ldots, g_q, x_1, \ldots, x_r) = 1$$
$$v_j(g_1, \ldots, g_q, x_1, \ldots, x_r) \neq 1$$

of equations and in-equations in variables x_i and the group elements g_j which has a solution in a group $H \geq G$ already has a solution in G.

Existentially Closed Groups

Examples of equations and inequations.

$$x^7 = g \quad \text{or } g^{-1}x_1^4 g_2 x_3^2 g_3^{10} = 1 \text{ or}$$

$$g^{-1}x_1^{-1}x_2^{-1}x_1x_2 = 1, \quad g_0^{-3}x_1^2 g_1^5 x_2^{-4} x_3^2 g_2 = 1$$

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are equations and

$$x_1^{-1}g_2x_6^{-1}x_1g_9x_5 \neq 1 \quad \text{and } x^{121}g^{-1} \neq 1$$

are in-equations with coefficients from the group G where $g_i, g \in G$.

κ-Existentially Closed Groups

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As an example; let a and b be two elements of different orders, say n and m. Then the equation $x^{-1}ax = b$ has no solution in any overgroup containing a and b.
Similarly

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As an example; let a and b be two elements of different orders, say n and m. Then the equation $x^{-1}ax = b$ has no solution in any overgroup containing a and b.
Similarly

$$x^5 = 1,\ x^4 = 1,\ x \neq 1$$

the system has no solution over any group.

Existentially Closed Groups

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After Scott's paper algebraically closed groups are studied by many group theorists, nowadays they are called as **existentially closed groups.**

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In particular there are existentially closed groups of any given infinite cardinality.

But we will see in coming slides that, this is not true for κ-existentially closed groups.

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Theorem 2 (B. H. Neumann [1])

There are $2^{\aleph_0}$ pairwise non-isomorphic 2-generator groups.

Using this result B. H. Neumann proved:

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The following beautiful argument is due to B. H. Neumann.

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But each countable group contains only countably many pairs of elements and thus only countably many 2-generator groups.

Hence $2^{\aleph_0}$ countable existentially closed groups are needed to accommodate all 2-generator groups.

Existentially Closed Groups

Theorem 4 (B. H. Neumann)

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Theorem 5 (B. H. Neumann)

An existentially closed group can not be finitely generated.

The proof of this is quite interesting and short. Let's have a look at the proof.

Existentially Closed Groups

Proof.

Let G be an existentially closed group. Let $X = \{g_1, g_2, \ldots g_n\}$ be a finite subset of G. We can solve the equations

$$x^{-1}g_1x = g_1, x^{-1}g_2x = g_2, x^{-1} \ldots = x^{-1}g_nx = g_n, x \neq 1$$

in the direct product $G \times H$ where H is a non-trivial group and hence in G.

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So G is not finitely generated. □

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κ-existentially closed groups are the analogs of existentially closed groups, allowing the number of equations and the number of in-equations to be infinite.

κ-Existentially Closed Groups

Definition. Let κ be an infinite cardinal. A group G with $|G| \geq \kappa$ is called κ-**existentially closed** if every system of less than κ-many equations and in-equations with coefficients in G which has a solution in some overgroup $H \geqslant G$ already has a solution in G.

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Lemma 6 (Kegel-K, 2018)

If κ is uncountable and G is a κ-existentially closed group, then isomorphic copy of every group A of order $|A| < \kappa$ is contained in G.

Moreover if κ is uncountable, then isomorphic copy of every group of order κ is contained in G.

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Proposition 7 (Kegel-K, 2018)

Let G be a group and κ be an uncountable cardinal. Then G is κ-existentially closed if and only if
(i) *G contains an isomorphic copy of every group of cardinality less than κ, and*
(ii) *every isomorphism between two subgroups of G of cardinality less than κ is induced by an inner automorphism of G.*

Existence of Explicit Examples of Existentially Closed Groups

B. H. Neumann in [4] stated that "However, no algebraically closed group is explicitly known, the existence proof being highly non-constructive. This stem in part from the fact that there is no useful criterion known that tells one what sentences are or are not consistent over a given group".

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In this talk, we will give [1], explicit examples of existentially closed groups for large cardinals. In particular we answer the more general question; existence of explicit examples of κ-existentially closed groups. Hence we answer Neumann's question in a more general case, positively.

Cayley's Theorem

The main ingredient of the construction of an existentially closed groups is the Cayley's Theorem.

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Observe that the above Lemma implies that the image under r of any two elements of the same order are conjugate in the symmetric group, $Sym(G)$.

But as you know; inside the symmetric group any two elements of order 2 may not be conjugate.

Cayley's Theorem

Indeed

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One may observe that the image of these elements under right regular representation r are conjugate.

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Let κ be any infinite regular cardinal. We may start with an arbitrary group G_0 of countably infinite order. Embed G_0 into $Sym(G_0) = G_1$ by right regular representation. Then embed G_1 into $Sym(G_1) = G_2$ again by right regular representation, continue like this, for limit ordinals β let $G_\beta = \bigcup_{i<\beta} G_i$. We continue until we reach the group G_κ.

Then the group G_κ is κ-existentially closed.

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The reason is the following Proposition:

κ-Existentially Closed Groups

Proposition 9 (Kegel-K, 2018)

Let G be a group and κ be an uncountable cardinal. Then G is κ-existentially closed if and only if
(i) *G contains an isomorphic copy of every group of cardinality less than κ, and*
(ii) *every isomorphism between two subgroups of G of cardinality less than κ is induced by an inner automorphism of G.*

Construction of explicit example of existentially Closed Groups

Since every κ-existentially closed group is an $\aleph_0$-existentially closed group, the above examples of κ-existentially closed groups are examples of $\aleph_0$-existentially closed groups.
This answers the B. H. Neumann's question positively.

Existence of κ-existentially closed groups of cardinality $\lambda \geq \kappa$

Corollary 10 (GCH)

Let $\lambda \geq \kappa$ be uncountable cardinals. Then there exists a κ-existentially closed group of cardinality λ if and only if $cf(\lambda) \geq \kappa$.

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In particular, if λ is a successor cardinal, then there exists a κ-existentially closed group of cardinality λ.

Moreover there exists no κ-existentially closed group of cardinality κ for singular cardinals.

Uniqueness

So by the above Corollary, we determine for which cardinals $\lambda \geq \kappa$, there exists κ-existentially closed groups of cardinality λ .

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We prove in [2, Theorem 2.7] that for an uncountable κ, any two κ-existentially closed groups of cardinality κ are isomorphic.

Uniqueness

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What can we say about the uniqueness of such groups?

We prove in [2, Theorem 2.7] that for an uncountable κ, any two κ-existentially closed groups of cardinality κ are isomorphic.

This is one of the differences between $\aleph_0$-existentially closed groups and κ-existentially closed groups.
As B. H. Neumann proved there are $2^{\aleph_0}$-existentially closed countable groups.

Open Question

Open Question Let κ be not an inaccessible cardinal. Does there exist an explicit example of κ-existentially closed group of cardinality κ?

Automorphisms of κ-existentially closed groups

Question What can we say about the automorphism groups of κ-existentially closed groups?

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We have seen in the explicit examples of κ-existentially closed groups that the construction has κ levels.

Definition. Let G be a group. An automorphism $\varphi \in Aut(G)$ is called κ-*inner* if for every subgroup $X \subseteq G$ with $|X| < \kappa$, there exists an element $g \in G$ such that $\iota_g(x) = \varphi(x)$ for all $x \in X$.

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Let κ-$Inn(G)$ denote the set of all κ-inner automorphisms of G.

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We clearly have

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Moreover, the inclusion on right is indeed an equality for κ-existentially closed groups.

Automorphisms of κ-existentially closed groups

Proposition 11

Let κ be uncountable and let G be κ-existentially closed. Then every automorphism of G is κ-inner.
i.e. $\kappa\text{-}Inn(G) = Aut(G)$.

We now introduce the notion of a level preserving automorphism. Let $C \subseteq \kappa$. An automorphism $\varphi \in Aut(G)$ is said to be C-*level preserving* if

$$\varphi[G_\alpha] = G_\alpha$$

for all $\alpha \in C$.

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We clearly have $Aut_\emptyset(G) = Aut(G)$ and

$$Aut_C(G) \leqslant Aut_D(G) \text{ whenever } D \subseteq C$$

Automorphisms of κ-existentially closed groups

Corollary 12

Let κ be inaccessible and let K be the unique κ-existentially closed group of cardinality κ, which (necessarily) is a limit of regular representations of length κ with countable base. Then

$$Aut(K) = \bigcup_{\substack{C \subseteq \kappa \\ C \text{ is club}}} Aut_C(K) = \bigcup_{\alpha < \kappa} Aut_{\{\alpha\}}(K)$$

Automorphisms of κ-existentially closed groups

Corollary 13

Let κ be inaccesible and let K be the unique κ-existentially closed group of cardinality κ. Then we have
$|Aut(K)| = 2^{\kappa}$.

THANK YOU

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