Graphs associated with Groups

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March 25, 2021

24 Hours of Ischia

Virtual Conference

Introduction

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I.e., each edge is between two distinct points.

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I should note that I will not be talking about the most well-known graph associated with groups: the Cayley graph.

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In the literature, nilpotent groups have also been considered.

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The *Commuting graph* of G is the graph with vertex set

$G \setminus Z(G)$ with an edge between x and y if $xy = yx$.

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is related to simple groups.

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This research culminated in with a paper by Solomon and Woldar where they prove that if S is a nonabelian simple group and G is any group and S and G have isomorphic commuting graphs, then S and G are isomorphic.

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In a seminal paper, Parker showed that if G is a solvable group with $Z(G) = 1$, then the commuting graph of G is disconnected if and only if G is a Frobenius group or a 2-Frobenius group.

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all have diameters at most 10.

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much less clear and even more wide open.

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We use $\Gamma(G)$ to denote the commuting graph of G.

Theorem 1 (B,C,C,H,L,L,P).

Let G be a group, let $Z = Z(G)$, and suppose that $G' \cap Z = 1$.

1. *$\Gamma(G)$ is connected if and only if $\Gamma(G/Z)$ is connected.*
2. *Every connected component of $\Gamma(G)$ has diameter at most 10.*
3. *If G is solvable and $\Gamma(G)$ is connected, then $\Gamma(G)$ has diameter at most 8.*
4. *If G is solvable, then $\Gamma(G)$ is disconnected if and only if G/Z is either a Frobenius group or a 2-Frobenius group.*

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$$C(G) \cap Z(G) = \{1\}.$$

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Theorem 2 (B,C,C,H,L,L,P).

If G is a group where $G/Z(G)$ is either a Frobenius or a 2-Frobenius group, then $\Gamma(G)$ is disconnected.

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This is the graph whose vertex set is $G \setminus \{1\}$ and there is an edge between x and y if $\langle x, y \rangle$ is cyclic.

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for the cyclic graph.

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the group is partitioned by cyclic subgroups.

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Lemma 3.

If G is a p-group for some prime p, then the number of connected components equals the number of subgroups of order p.

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Theorem 4 (C,L,S,T,U).

If G is nilpotent and $|G|$ is divisible by at least two primes, then $\Delta(G)$ is connected with $\mathrm{diam}(\Delta(G)) \leq 3$.

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In fact, we can determine exactly which nilpotent groups have

cyclic graphs of diameter 2 and which have diameter 3.

Lemma 5 (C,L,S,T,U).

Let G be a nilpotent group that does not have prime power order. Then the following are true:

1. *If at least one but not all Sylow subgroups are cyclic or generalized quaternion, then $\Delta(G)$ has diameter 2.*
2. *If no Sylow subgroup is cyclic or generalized quaternion, then $\Delta(G)$ has diameter 3.*

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We consider the cyclic graphs of nontrivial direct products.

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Lemma 6 (C,L,S,T,U).

If G and H are nontrivial groups with coprime orders, then $\Delta(G \times H)$ is connected with $\mathrm{diam}(\Delta(G \times H)) \leq 3$.

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Theorem 7 (C,L,S,T,U).

If G and H are nontrivial groups and the graph $\Delta(G \times H)$ is connected, then $\operatorname{diam}(\Delta(G \times H)) \leq 7$.

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Then $\Delta(G \times H)$ has diameter 7.

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extracted.

Theorem 8 (C,L,S,T,U).

If G and H are groups with $\operatorname{diam}(\Delta(G \times H)) \leq 2$, *then*

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Theorem 9 (C,L,S,T,U).

Let G and H be nontrivial groups. The graph $\Delta(G \times H)$ is disconnected if and only if $G \times H$ satisfies $(\mathcal{C}_1(p))$ for some prime p.

Corollary 10 (C,L,S,T,U).

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Corollary 10 (C,L,S,T,U).

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Theorem 11 (C,L,S,T,U).

Let G be a Z-group. Then $\Delta(G)$ is disconnected if and only if G is a Frobenius group.

We also prove:

Theorem 12 (C,L,S,T,U).

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Theorem 13 (C,L,S,T,U).

If G is a Z-group, then $\operatorname{diam}(\Delta(G)) \leq 2$ *if and only if* $Z(G) \neq 1$.

We provide examples of Z-groups with diameters 2, 3, and 4.

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Theorem 14 (C,L,S,T,U).

Let p and q be distinct primes, and let G be a $\{p, q\}$-group. Then, $\operatorname{diam}(\Delta(G)) = 2$ *if and only if G has a unique subgroup of order p or a unique subgroup of order q and that subgroup is central in G.*

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Theorem 15 (C,L,S,T,U).

If G is a $\{p, q, r\}$-group and the cyclic graph of G has diameter 2, then $Z(G) > 1$.

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then $Z(G) = 1$. It follows that the cyclic graph and the commuting

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When G is a Frobenius group with Frobenius kernel N, it is not difficult to see that the number of connected components in the commuting graph of G is $1 + |N|$.

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G/K are Frobenius groups with Frobenius kernel K and L/K, When G is a 2-Frobenius group where $K \leq L$ satisfies that L and G/K are Frobenius groups with Frobenius kernel K and L/K, the commuting graph of G has $1 + |K|$ connected components.

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Set $m_p(G)$ to be the number of subgroups

of order p. For Frobenius groups, we obtain:

Theorem 16 (C,L).

Let G be a Frobenius group with Frobenius kernel N. If N is a p-group for some prime number p, then $\Delta(G)$ has $|N| + m_p(N)$ connected components. If N is not a group of prime power order, then $\Delta(G)$ has $|N| + 1$ connected components.

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Theorem 17 (C,L).

Let G be a 2-Frobenius group with K as in the definition. If $|K|$ is divisible by at least two distinct prime numbers, then $\Delta(G)$ has $|K| + 1$ connected components.

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Theorem 18 (C,L).

Let G be a 2-Frobenius group, and assume that K and G/L are p-groups for some prime p, where K and L are as in the definition. Then $\Delta(G)$ has $|K| + m_p(G)$ connected components.

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Theorem 19 (C,L).

Let G be a 2-Frobenius group, and let p be a prime number. Assume that K is a p-group for some prime p and that G/L is not a p-group, where K and L are as in the definition. Then the number of connected components of $\Delta(G)$ is

$$|K| + |K : L| + m_p^*,$$

where m_p^* is the number of subgroups of order p in G that are not centralized by an element of prime order other than p.

We switch gears again.

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throw out any universal vertices.

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Because of this, most of the results in the literature look at the set

$G \setminus \{1\}$ as the vertices of the cyclic graph.

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universal vertices for this graph.

Theorem 20 (C,L,S,T,U).

Let G be a group, $g \in G$, and $\pi = \pi(o(g))$. Write $g = \prod_{p \in \pi} g_p$, where each g_p is a p-element for $p \in \pi$ and $g_p g_q = g_q g_p$ for all $p, q \in \pi$. Then g is a universal vertex for $\Delta(G)$ if and only if, for each $p \in \pi$, a Sylow p-subgroup P of G is cyclic or generalized quaternion and $\langle g_p \rangle \leq P \cap Z(G)$.

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Let G be a group, for an element $x \in G$, define

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(Note that in the commuting graph, the set of neighbors of x is

the set $C_G(x) = \{y \in G \mid xy = yx\}$, which is a subgroup of G.)

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Theorem 21 (B,C,C,H,L,L,P).

Let G be a p-group for some prime p. Then G is tidy if and only one of the following occurs:

1. *G has exponent p.*
2. *G is cyclic.*
3. *$p = 2$ and G is dihedral or generalized quaternion.*

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Hence, the Sylow subgroups of a tidy group are all tidy.

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Theorem 22 (B,C,C,H,L,L,P).

Suppose G is a solvable group and let π be the set of primes dividing $|G|$. If G has a tidy Hall ρ-subgroup for each subset $\rho \subseteq \pi$ of size 2, then G is tidy.

We can classify the tidy $\{p, q\}$-groups:

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Theorem 23 (B,C,C,H,L,L,P).

Suppose G is a $\{p, q\}$-group for distinct primes p and q. Then G is tidy if and only if G has tidy Sylow p- and Sylow q-subgroups and one of the following occurs:

1. *G is nilpotent.*
2. *Up to relabeling p and q, Z_∞ is a q-group and G/Z_∞ is a Frobenius group whose Frobenius kernel is the Sylow p-subgroup.*
3. *$\{p, q\} = \{2, 3\}$, $O_2(G)$ is a Klein 4-group, $G/O_3(G) \cong S_4$ and $G/O_2(G)$ is a Frobenius group whose Frobenius kernel is the Sylow 3-subgroup of $G/O_2(G)$ and whose Frobenius complement has order 2. Also, $Z(G) = 1$.*

Theorem (Continued).

4 $\{p, q\} = \{2, 3\}$, $O_2(G)$ *is a Sylow* 2*-subgroup of* G *and is the quaternion group of order* 8, $G/O_3(G) \cong \mathrm{SL}_2(3)$. *Also,* $Z_\infty = Z(O_2(G)) \times O_3(G)$.

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5. $\{p,q\} = \{2,3\}$, $O_2(G)$ *is the quaternion group of order* 8, $G/O_3(G) \cong \widetilde{\mathrm{GL}_2(3)}$ *and* $G/O_2(G)$ *is a Frobenius group whose Frobenius kernel is the Sylow* 3*-subgroup of* $G/O_2(G)$ *and whose Frobenius complement has order* 2. *Also,* $Z_\infty = Z(G) = Z(O_2(G))$.

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Theorem 24 (B,C,C,H,L,L,P).

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Theorem 25 (B,C,C,H,L,L,P).

Let G be a solvable, tidy group. Then G has Fitting height at most 4 and $G/F(G)$ has derived length at most 4. If $|G|$ is odd, then G has Fitting height at most 3 and $G/F(G)$ is abelian or metabelian.

Solvable (Soluble) graph

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This is the largest normal subgroup of G that is solvable.

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Theorem 26 (A,L,M,M).

For every group G, the solubility graph $\Delta_{\mathcal{S}}(G)$ is connected, and its diameter is at most 11.

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It is not difficult to see that $\mathrm{Sol}_G(x)$ is not necessarily a subgroup. In A_5, this set is a subgroup when x has order 5 and is not a subgroup otherwise.

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Theorem 27 (A,L,M,M).

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Theorem 28 (A,L,M,M).

Let G be a group. If there exists $x \in G$ so that the elements of $\mathrm{Sol}_G(x)$ commute pairwise, then G is abelian.

We also obtained the following:

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Theorem 29 (A,L,M,M).

Let G be a group. The following are equivalent:

1. *G is soluble.*
2. *For each conjugacy class $\mathcal{C}$ of G, the induced subgraph $\Gamma_{\mathcal{S}}(\mathcal{C})$ is a clique.*
3. $\mathrm{Sol}_G(x) \cap \mathcal{C} \neq \emptyset$ *for every $x \in G$ and every conjugacy class $\mathcal{C}$ of G.*

Thank You!

Questions?

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However, if G_1 and G_2 are isomorphic, then they are isoclinic.

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$$[\sigma(aZ(G_1)), \sigma(bZ(G_1))] = \tau([a, b]) \text{ for all } a, b \in G_1.$$

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Define the graph $C^*(G)$ to be the graph obtained by taking the subgraph of $C(G)$ induced by a transversal for $Z(G)$ in G.

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It is not difficult to see that $C^*(G)$ is independent of the transversal chosen since a and b commute if and only if az_1 and bz_2 commute for all $z_1, z_2 \in Z(G)$.

It is immediate to see that if G_1 and G_2 have isomorphic commuting graphs, then $C^*(G_1) \cong C^*(G_2)$.

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With this in mind, we then see that when G_1 and G_2 are isoclinic with $|G_1|$ and $|G_2|$, then G_1 and G_2 have isomorphic commuting graphs.

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Probably not,

but we would be very interested to see a counterexample.

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The following two facts relate these sets.

Lemma 30.

Let G be a group. If $Z \in \mathcal{Z}(G)$ and $C = C_G(Z)$, then $C \in \mathcal{C}(G)$ and $Z = Z(C)$. In particular, the maps $C \mapsto Z(C)$ from $\mathcal{C}(G) \to \mathcal{Z}(G)$ and $Z \mapsto C_G(Z)$ from $\mathcal{Z}(G)$ to $\mathcal{C}(G)$ are inverse maps, and thus, bijections.

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Lemma 31.

Let G be a group and suppose $a, b \in G \setminus Z(G)$.

1. *If $a \in C_G(b)$, then $Z(a) \leq C_G(b)$.*
2. *$Z(a) \leq C_G(b)$ if and only if $Z(b) \leq C_G(a)$.*

We let $\Gamma_{\mathcal{Z}}(G)$ be the graph with vertices $\{Z \in \mathcal{Z}(G)\}$.

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If $Z_1, Z_2 \in \mathcal{Z}(G)$ with $Z_1 \neq Z_2$, then there

is an edge between Z_1 and Z_2 precisely when $Z_2 \leq C_G(Z_1)$.

Notice via Lemma 31 that $Z_2 \leq C_G(Z_1)$ if and only if

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$Z_1 \leq C_G(Z_2)$. Hence, it really does make sense to think

of this as an undirected graph. Recall that $\mathcal{C}(G)$ is in bijection

with $\mathcal{Z}(G)$, so we could have used $\{C \in \mathcal{C}(G)\}$ for our vertex set.

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then we use $N(u)$ to denote the neighbors of u.

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I.e., $N(u)$ is the set of vertices in Γ that are adjacent to u.

We define an equivalence relation on the vertices of Γ.

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We say that $u \sim v$ if either $u = v$ or u is adjacent to v and

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$\{u\} \cup N(u) = \{v\} \cup N(v)$.

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The vertices of this graph are the equivalence classes under $\sim$.

If $[u]$ and $[v]$ are the equivalence classes of u and v, then $[u]$ and

If $[u]$ and $[v]$ are the equivalence classes of u and v, then $[u]$ and

$[v]$ are adjacent in $\Gamma/\sim$ if and only if u and v are adjacent

If $[u]$ and $[v]$ are the equivalence classes of u and v, then $[u]$ and $[v]$ are adjacent in $\Gamma/\sim$ if and only if u and v are adjacent in Γ. Observe that $\sim$ is uniquely determined by Γ.

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We show that $\Gamma_{\mathcal{Z}}(G)$ can be obtained from the commuting

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We show that $\Gamma_{\mathcal{Z}}(G)$ can be obtained from the commuting

graph of G and $C^*(G)$ via this equivalence relation.

We use $C(G)$ to denote the commuting graph of G.

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Lemma 32.

Let G be a group. Then the map $Z(g) \mapsto [g]$ is a graph isomorphism from $\Gamma_{\mathcal{Z}}(G)$ to $C(G)/\sim$ or $C^*(G)/\sim$.

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This implies that $\Gamma_{\mathcal{Z}}(G)$ and $C(G)$ have the same number of

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This implies that $\Gamma_{\mathcal{Z}}(G)$ and $C(G)$ have the same number of

connected components and that the diameters of the

corresponding components are the same with one exception.

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and the corresponding component in $C(G)$ will be complete.

Lemma 33.

Let G be a group, and $g, h \in G \setminus Z(G)$. If $C_G(g) \cap C_G(h) > Z(G)$, then $Z(g)$ and $Z(h)$ have distance at most 2 in $\Gamma_{\mathcal{Z}}(G)$. Equivalently, g and h have distance at most 2 in $C(G)$ and $gZ(G)$ and $hZ(G)$ have distance at most 2 in $C^*(G)$.

Lemma 33.

Let G be a group, and $g, h \in G \setminus Z(G)$. If $C_G(g) \cap C_G(h) > Z(G)$, then $Z(g)$ and $Z(h)$ have distance at most 2 in $\Gamma_{\mathcal{Z}}(G)$. Equivalently, g and h have distance at most 2 in $C(G)$ and $gZ(G)$ and $hZ(G)$ have distance at most 2 in $C^*(G)$.

Theorem 34.

Let G be a group. If $|G'| < |G : Z(G)|^{1/2}$, then $C(G)$ is connected and has diameter at most 2.

We now characterize the isolated vertices in $\Gamma_{\mathcal{Z}}(G)$.

Lemma 35.

Let G be a group. Let $g \in G \setminus Z(G)$. Then the following are equivalent:

1. $C_G(g)$ *is abelian and for all* $h \in G \setminus Z(G)$, *either* $C_G(h) = C_G(g)$ *or* $C_G(h) \cap C_G(g) = Z(G)$.
2. $C_G(h) = C_G(g)$ *for all* $h \in C_G(g) \setminus Z(G)$.
3. $Z(h) = Z(g)$ *for all* $h \in C_G(g) \setminus Z(G)$.
4. $Z(h) = C_G(g)$ *for all* $h \in C_G(g) \setminus Z(G)$.
5. $Z(g)$ *is an isolated vertex in* $\Gamma_{\mathcal{Z}}(G)$.

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Hence, Z is an isolated vertex in $\Gamma_{\mathcal{Z}}(G)$ if and only

if $C_G(Z)$ is abelian and maximal among the subgroups in $\mathcal{C}(G)$.

Recall that an empty graph is a graph with no edges.

One consequence of Lemma 35 is that if $\Gamma_{\mathcal{Z}}(G)$ is

an empty graph, then $C_G(x)$ is abelian for all $x \in G \setminus Z(G)$.

A group G is called a *CA-group* if $C_G(x)$ is abelian for all

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We claim that if G is a CA-group, then $\Gamma_{\mathcal{Z}}(G)$ is empty.

Corollary 36.

Let G be a group. Then $\Gamma_{\mathcal{Z}}(G)$ is an empty graph if and only if G is a CA-group.

Lemma 37.

Let G be a group. Let $g \in G \setminus Z(G)$. The following are equivalent:

1. *For all $h \in G \setminus Z(G)$, either $C_G(h) \leq C_G(g)$ or $C_G(h) \cap C_G(g) = Z(G)$.*
2. *$C_G(h) \leq C_G(g)$ for all $h \in C_G(g) \setminus Z(G)$.*
3. *$Z(g) \leq Z(h)$ for all $h \in C_G(g) \setminus Z(G)$.*
4. *$C_G(g) \setminus Z(G)$ is a connected component in $\mathfrak{C}(G)$.*