Generalized nilpotency properties for subgroup lattices of groups

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"Solving a problem without telling it, without discussing its nature with a friend, means losing most of the taste for Mathematics" (Ennio De Giorgi)

- The central theme of the lattice group theory is the relation between the structure of a group and the structure of its lattice of subgroups
- Many group theorists have worked in this area: Baer, Sadovskii, Suzuki, Zacher, Schmidt
- There have been many contributions from the whole Neapolitan school, starting with Mario Curzio and then his pupils

We denote by $L(G)$ the subgroup lattice of a group G

- Given a class of groups X, what can we say about the subgroup lattices of groups in X?
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The oldest result of this type is the following beautiful

Theorem (Ore 1938). The group G is locally cyclic if and only if $L(G)$ is distributive.

It follows that a group G is cyclic if and only if $L(G)$ is distributive and it satisfies the maximal condition

If G and $\bar{G}$ are groups, an isomorphism from $L(G)$ in $L(\bar{G})$ is called a projectivity from G to $\bar{G}$

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There are important classes of groups that are not invariant under projectivities
The elementary abelian group of order 9 and the nonabelian group of order 6 have isomorphic subgroup lattices

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for instance: finite group by finite lattice

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At least for finite groups, modularity is a good translation of normality .

Theorem (R. Schmidt, 1968) A finite group G is soluble if and only if there exists a chain

$$\{1\} = M_0 \leqslant M_1 \leqslant \cdots \leqslant M_n = G$$

of modular subgroups of G such that $[M_{i+1}/M_i]$ is modular, for every $i < n$

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Theorem (R. Schimdt, 1969) If M is a maximal proper modular subgroup of a finite group G, then either M is normal in G or G/M_G is nonabelian of order pq for primes p and q.

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The situation is much more complicated for infinite groups

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A permodular subgroup M is a modular subgroup with the following additional property:
(*) for every cyclic subgroup X in G and $H \leqslant G$ such that $M \leqslant H \leqslant < M, X >$ and the interval $[< M, X > /H]$ is finite, then the index $| < M, X >: H|$ is finite

Since cyclic subgroups of G (by Ore's theorem) and the finiteness of the index of a subgroup (by a theorem of Zacher-Rips) can be recognized in the subgroup lattice, then (*) is a lattice-theoretic property.

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Even every permutable subgroup X of a group G (i.e. a subgroup such that $XH = HX$ for every subgroup H of G) is permodular

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$L(G)$ is permodular if and only if every subgroup of G is permodular

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The first to use this idea, in 1954, were Kontorovič and Plotkin to introduce the concept of a permodular chain as translation of central series of groups

The definition of permodular chain is done in a general (complete) lattice and it is quite technical. For groups, substantially we have:

a permodular chain of a group G is a chain

$$\{1\} = M_0 \leqslant M_1 \leqslant \cdots \leqslant M_n = G$$

of permodular subgroups of G such that for any $x \in G$ the interval $[< M_{i+1}, x > /M_i]$ is permodular for every $i < n$ (M_{i+1} is permodularly embedded in $[G/M_i]$, for every $i < n$)

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Kontorovič and Plotkin used this notion to obtain a lattice-theoretic characterization of the class of torsion-free nilpotent groups

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for instance to Wielandt's theorem: every maximal subgroup of a finite group G is normal if and only if G is nilpotent

Theorem (R. Schmidt, 1970) Every maximal subgroup of a finite group G is modular if and only if G is supersoluble and induces an automorphism group of order 1 or prime in every complemented chief factor of G

Clearly, for a finite group G also the condition that every maximal subgroup of every subgroup H is normal in H, is equivalent to nilpotency.

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The groups with that property are called lower semimodular and were characterized by Ito

Theorem (Ito, 1951) A finite group G is lower semimodular if and only if G is supersoluble and induces an automorphism group of order 1 or prime in every chief factor of G.

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Theorem (R. Schimdt, 2013) If G is a finite group with a modular chain, then G is lower semimodular.

A lattice translation of Schur theorem

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It follows that any projective image $\bar{G}$ of a central-by-finite group G contains a finite normal subgroup $\bar{N}$ with modular factor group

Groups with a n-centre of finite index

Let G be a group such that a term $Z_n(G)$ of the upper central series has finite index in G, for some integer n

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(B) There exists a finite chain

$$\{1\} = M_0 \leqslant M_1 \leqslant \cdots \leqslant M_n$$

of permodular subgroups of G such that each M_{i+1} is permodularly embedded in the interval $[G/M_i]$ and M_n has finite index in G.

Theorem (M. De Falco, F. de Giovanni, C.M., 2020) Let G be a group satisfying the property (B). Then the following conditions hold:

(i) All periodic sections of G are locally finite.

(ii) The set T consisting of all elements of G of finite order is a subgroup.

(iii) The factor group G/T is nilpotent of class at most n.

It is easy to see that if $|G : Z_n(G)|$ is finite, G contains only finitely many maximal non-normal subgroups, since $Z_n(G)$ is contained in the intersection $\Delta(G)$ of all maximal non-normal subgroups of G. In particular $\Delta(G)$ has finite index in G.

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Denoted by $\Phi(G)$ the Frattini subgroup of a group G, we have:

Let G be a group. Then $\theta(G)/\Phi(G)$ is metabelian and for every complemented abelian chief factor H/K of G such that $H \leqslant \theta(G)$, the factor H/K has prime order and G induces on it a group of automorphisms of order 1 or prime.

If G has finitely many maximal non-modular subgroups, then $\theta(G)$ has finite index in G.

Theorem (M. De Falco, F. de Giovanni, C. M., 2020) A group G has finitely many maximal non-modular subgroups if and only if G contains a normal subgroup of finite index L satisfying the following conditions:

(1) $L/\Phi(G)$ is metabelian;

(2) for every complemented chief factor H/K of G such that $\Phi(G) \leqslant K$ and $H \leqslant L$, the factor H/K has prime order and G induces on it a group of automorphisms of order 1 or prime.

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Theorem (M. De Falco, F. de Giovanni, C. M., 2020) Let G be a finitely generated group satisfying the property (B). Then $\gamma_{n+1}(G)$ is finite.

As consequence, we have established the invariance under projectivities of the class of finitely generated groups with a n-th centre of finite index.

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As consequence, we have established the invariance under projectivities of the class of finitely generated groups with a n-th centre of finite index.

Corollary Let G be a finitely generated group such that $G/Z_n(G)$ is finite. Then any projective image $\bar{G}$ of G has the same property.