Sylow branching coefficients and solvability of finite groups

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Z. Balanov, M. Muzychuk, H. Wu. On algebraic problems behind the Brouwer degree of equivariant maps, Journal of Algebra, 649 (2020), 45-77.

Main result

Definition (E. Giannelli)

Let P be a p-Sylow subgroup of a finite group G. The numbers $Z^{\chi}_{1_P} := (\chi, (1_P)^G)$ are called Sylow branching coefficients of $(1_P)^G$.

$$(1_P)^G = \sum_{\chi \in \mathrm{Irr}(G)} Z^{\chi}_{1_P} \chi, \quad \mathrm{Irr}((1_P)^G) := \{\chi \in \mathrm{Irr}(G) \mid Z^{\chi}_{1_P} \neq 0\}.$$

Theorem (Balanov, M. and Wu)

A finite group is solvable iff

$$\bigcap_{p \in \pi(G)} \mathrm{Irr}((1_P)^G) = \{1_G\} \iff \forall_{\chi \in \mathrm{Irr}^*(G)} \exists_{p \in \pi(G)} : \chi \notin \mathrm{Irr}((1_P)^G).$$

α-characteristic of a linear representation

Let G be a finite group,
$\mathrm{IRR}(G)$ the set of all complex irreducible representations of G,
$\mathrm{Irr}(G)$ the set of all complex irreducible characters of G,
$\mathrm{IRR}^*(G) = \mathrm{IRR}(G) \setminus \{1_G\}$, $\mathrm{Irr}^*(G) = \mathrm{Irr}(G) \setminus \{1_G\}$.

Definition (Z. Balanov & A. Kushkuley)

Let $\rho : G \to GL(V)$ be f.d. complex linear representation of G. The greatest common divisor of lengths of G-orbits in its action on $V \setminus \{0\}$ is called the α-characteristic of ρ; $\alpha(\rho)$ is trivial iff $\alpha(\rho) = 1$.

Trivial observation

$\alpha(\rho)$ divides $|G|$.

Motivation

Definition

Let $\rho : G \to GL(V), \sigma : G \to GL(W)$ be representations of G. A function $\Phi : V \to W$ is called G-equivariant iff

$$\forall_{g \in G} \forall_{v \in V} \ \Phi(\rho(g)v) = \sigma(g)\Phi(v).$$

Congruence Principle (after Z. Balanov and A. Kushkuley, 1996)

Let V, W be two unitary n-dimensional G-representations. Assume that there exists a G-equivariant map $\Phi : V \to W \setminus \{0\}$. Then, for **any** equivariant map $\Psi : V \to W \setminus \{0\}$, one has

$$\deg(\Psi) \equiv \deg(\Phi) (\mathrm{mod}\ \alpha(V)).$$

Motivation

The congruence principle can be effectively applied only if there exists at least one equivariant map $\Phi : V \to W \setminus \{0\}$ with $\deg(\Phi)$ easy to calculate. For example, if $W = V$, then one can take $\Phi = Id_V$, in which case,

$$\deg(\Psi) \equiv 1 (\operatorname{mod} \alpha(V)).$$

for every equivariant map $\Psi : V \to V \setminus \{0\}$.

Problem A

Under which conditions on a G-representation V, is $\alpha(V)$ greater than 1?

Problem B

Under which conditions on V and W, does there exist an equivariant map $\Phi : V \to W \setminus \{0\}$ with $\deg(\Phi)$ easy to calculate?

How to compute the α-characteristic

Let V be an $\mathbb{C}[G]$-module. Given a non-zero $v \in V$, a conjugacy class of the point stabilizer G_v is called the orbit type of v. Define

$$\Phi(G, V) := \{(G_v) \mid v \in V^*\}.$$

Proposition

$$\alpha(V) = \gcd\{[G : H] \mid (H) \in \Phi(G, V)\}.$$

Remark

$\alpha(V)$ cannot be computed from the character table of G. The quaternion group Q_8 and the dihedral group D_8 have the same character table, but distinct α-characteristics of their 2-dimensional modules.

Examples

$G = S_4$, $V = \{(v_1, ..., v_4) \mid v_1 + ... + v_4 = 0\}$. Let $v = (v_1, ..., v_4) \in V$ be an arbitrary vector. Then

- $G_v = 1$ if all v_i's are pairwise distinct;
- $G_v \cong \mathbb{Z}_2$ iff $|\{v_1, ..., v_4\}| = 3$;
- $G_v \cong S_3$ iff three of v_i's are equal;
- $G_v \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ iff v is of type $\{x, x, -x, -x\}$.

Thus $\alpha(V) = 2$.

	V_0	V_1	V_2	V	$V_1 \otimes V$
dim	1	1	2	3	3
α	1	2	3	2	4

Examples

The group $G = A_5$ has 5 irreducible representations: V_0, V_1, V_2, V_3, V_4 which have the following parameters.

	dim	$\Phi(G, V)$	$\alpha(V)$
V_0	1	$\{(A_5)\}$	1
V_1	3	$\{(\mathbb{Z}_2), (\mathbb{Z}_3), (\mathbb{Z}_5)\}$	2
V_2	3	$\{(\mathbb{Z}_2), (\mathbb{Z}_3), (\mathbb{Z}_5)\}$	2
V_3	4	$\{(\mathbb{Z}_2), (\mathbb{Z}_3), (A_4), (S_3)\}$	5
V_3	5	$\{(\mathbb{Z}_2), (S_3), (\mathbb{Z}_2 \times \mathbb{Z}_2), (D_{10})\}$	1

Questions

What can we say about a group G if all its non-trivial irreducible complex representations have
(a) non-trivial α-characteristic?
(b) trivial α-characteristic?

Some properties of α-characteristic

Proposition

Let U, V be $\mathbb{C}[G]$-modules. Then

- $\alpha(U \oplus V) = \gcd(\alpha(U), \alpha(V))$;
- $\alpha(U \otimes V)$ divides $\alpha(U)\alpha(V)$;
- If U and V are Galois-conjugate, then $\alpha(U) = \alpha(V)$;
- If $U = V^\sigma$ for some $\sigma \in \mathrm{Aut}(G)$, the $\alpha(U) = \alpha(V)$.

The division in the second line could be proper. For example, if $G = S_3$ and $U = V$ is the unique irreducible 2-dimensional representation of S_3.

Induction and restriction

Proposition

Let H be a subgroup of G, V an $\mathbb{C}[G]$-module and U an $\mathbb{C}[H]$-module. Then

- $\alpha(V_H)$ divides $\alpha(V)$;
- If $H \trianglelefteq G$ then $\alpha(U)$ divides $\alpha(U^G)$.

If H is not normal in G then the second assertion is not true anymore. Example $G = S_3, H \cong S_2$ and U the unique non-trivial $\mathbb{C}[S_2]$-module.
The case when $\alpha(U) \neq \alpha(U^G)$ happens (for example, $G = Q_8, H \cong \mathbb{Z}_4$).

Corollary

If p is a prime divisor of $|G|$, then $\alpha(V)_p = \alpha(V_P)$ where $P \in \mathrm{Syl}_p(G)$;

Sylow branching coefficients and α-characteristic

Proposition

Let ρ be an irreducible representation of G, χ the character of ρ. Fix $p \in \pi(G)$ and $P \in \mathrm{Syl}_p(G)$. TFAE

- p divides $\alpha(\rho)$;
- 1_P is not a constituent of χ_P;
- $\chi \notin \mathrm{Irr}(1_P)^G$;
- $\rho(\underline{P}) = 0$.

Corollary

An irreducible representation ρ has trivial alpha characteristic iff $\chi_\rho \in \bigcap_{p \in \pi(G)} \mathrm{Irr}((1_P)^G)$.

Solvability Criterion

Theorem

Given a finite group G, TFAE

(1) G is solvable;

(2) α-characteristic of any non-trivial irreducible representation of G is non-trivial;

(3) $\bigcap_{p\in\pi(G)} \mathrm{Irr}((1_P)^G) = \{1_G\}$.

Proof of SC: (1) $\Longrightarrow$ (2)

Proposition

Let $\rho : \mathbb{C}[G] \to GL(V)$ be an irreducible representation. If $O_p(G/\ker(\rho)) \neq 1$ for some prime p, then p divides $\alpha(\rho)$.

Proof.

ρ is an irreducible faithful representation of $\bar{G} := G/\ker(\rho)$. The subspace $V^{O_p(\bar{G})}$ is $\bar{G}$-invariant $\Rightarrow$ either $V^{O_p(\bar{G})} = V$ or $V^{O_p(\bar{G})} = \{0\}$. The first case contradicts ρ being faithful. In the second one every non-zero $O_p(\bar{G})$-orbit has more than one element, and, therefore, its cardinality is divisible by p. Hence $\alpha(\rho_{O_p(\bar{G})}) = p^e, e > 0 \Rightarrow p^e \big| \alpha(\rho)$. □

Corollary

If G is solvable, then $\alpha(\rho) > 1$ for each $\rho \in \mathrm{IRR}^*(G)$.

Proof of SC: (2) $\Longrightarrow$ (1)

Definition

Let $\pi(G) = \{p_1, ..., p_k\}$ be the set of prime divisors of $|G|$. An ordered k-tuple $(P_1, ..., P_k)$ of Sylow subgroups $P_i \in \mathrm{Syl}_{p_i}(G)$ is called a complete system of Sylow subgroups.

Lemma

Let G be a finite group. Fix an ordering $p_1, ..., p_k$ of $\pi(G)$. TFAE

(a) $\forall_{\rho \in \mathrm{IRR}^*(G)} \exists$ a Sylow p – subgroup $P \leq G$ s.t. $\rho(\underline{P}) = 0$;

(b) $P_1 P_2 \cdots P_k = G$ for any complete system of Sylow subgroups $P_i \in \mathrm{Syl}_{p_i}(G)$.

(b') $\underline{P_1} \cdot \underline{P_2} \cdots \underline{P_k} = \underline{G}$ for any complete system of Sylow subgroups $P_i \in \mathrm{Syl}_{p_i}(G)$.

Proof of the Lemma: (a) $\implies$ (b')

Proposition

An element $x \in \mathbb{C}[G]$ belongs to the principal ideal generated by $\underline{G}$ iff $\rho(x) = 0$ holds for every $\rho \in \mathrm{IRR}^*(G)$.

Proof.

(a)$\Rightarrow$(b')
Denote $\Pi = \underline{P_1} \cdot \underline{P_2} \cdots \underline{P_k}$.
Pick an arbitrary $\rho \in \mathrm{IRR}^*(G)$. Then $\rho(\underline{P_i}) = 0$ for some i.
Therefore $\rho(\Pi) = 0$.
Thus $\rho(\Pi) = 0$ for every $\rho \in \mathrm{IRR}^*(G) \implies \Pi = \lambda \underline{G}$ for some $\lambda \in \mathbb{C}$. Applying 1_G to both sides we obtain $\lambda = 1$. $\square$

Proof of the Lemma:$(b') \Rightarrow (a)$

Assume, towards a contradiction, that

$$\exists \rho \in \mathrm{IRR}^*(G) \text{ s.t. } \rho(\underline{P_i}) \neq 0 \text{ for all } i = 1, ..., k.$$

Let ℓ be the smallest number with $\rho(\underline{P_1}) \cdots \rho(\underline{P_\ell}) = 0$ for all $P_i \in \mathrm{Syl}_{p_i}(G), i = 1, ..., \ell$. Clearly, $1 < \ell \leq k$. Then

$$\sum_{P_\ell \in \mathrm{Syl}_{p_\ell}(G)} \rho(\underline{P_1}) \cdots \rho(\underline{P_\ell}) = 0 \implies$$

$$\rho(\underline{P_1}) \cdots \rho(\underline{P_{\ell-1}}) \left(\sum_{P_\ell \in \mathrm{Syl}_{p_\ell}(G)} \rho(\underline{P_\ell}) \right) = 0.$$

Proof of the Lemma:$(b') \Rightarrow (a)$

Since $\frac{1}{|P_\ell|}\rho(\underline{P_\ell})$ is a non-zero idempotent, the trace of $\rho(\underline{P_\ell})$ is non-zero. Therefore the trace of

$$\sum_{P_\ell \in \mathrm{Syl}_{p_\ell}(G)} \rho(\underline{P_\ell}) = \rho(\sum_{P_\ell \in \mathrm{Syl}_{p_\ell}(G)} \underline{P_\ell})$$

is non-zero, implying that this sum is a non-zero scalar matrix. Hence $\rho(\underline{P_1}) \cdots \rho(\underline{P_{\ell-1}}) = 0$, contrary to minimality of ℓ. □

Theorem (P. Hall)

If G is solvable, then the product of any complete system of Sylow's subgroups is equal to G.

Theorem (G. Kaplan & D. Levy, 2005)

The product of any complete system of Sylow subgroups is equal to G iff G is solvable.

Non-solvable groups

Definition

Let us define $A(G)$ to be the intersections of all kernels $\ker(\rho)$ when ρ runs through all irreps of G with $\alpha(\rho) = 1$. Clearly, $A(G)$ is a characteristic subgroup of G.

Proposition

$\mathrm{Sol}(G) \leq A(G)$.

The equality holds if the answer on the following question is affirmative.

Question

Let $N \trianglelefteq G$ and $\theta \in \mathrm{IRR}(N)$ has trivial α-characteristic. Is it true that θ^G contains an irreducible constituent ρ with $\alpha(\rho) = 1$?

The answer is affirmative if θ^G is irreducible or G/N is solvable.

"Anti"-solvable groups

Definition

Let us call a group G "anti"-solvable if every irreducible representation of G has trivial α-characteristic.
Equivalently, $\bigcap_{p\in\pi(G)} \mathrm{Irr}((1_P)^G) = \mathrm{Irr}(G)$.
We denote the set of all such groups as $\mathfrak{T}$.

No solvable group is contained in $\mathfrak{T}$.

Example

The first Janko group $J_1 \in \mathfrak{T}$, $|J_1| = 2^3.3.5.7.11.19$. But $J_2 \notin \mathfrak{T}$.

"Anti"-solvable groups

Theorem

If $G \in \mathfrak{T}$ and $N \trianglelefteq G$, then $N, G/N \in \mathfrak{T}$. If $G, H \in \mathfrak{T}$ then $G \times H \in \mathfrak{T}$

Corollary

The composition factors of a $\mathfrak{T}$-group are non-abelian simple $\mathfrak{T}$-groups.

Sporadic simple groups

Among $M_{11}, M_{12}, J_1, M_{22}, J_2, M_{23}, HS, J_3, M_{24}, McL, He, Suz$ only J_1 and J_3 belong to $\mathfrak{T}$.

Conjecture

All "anti"-solvable simple groups are sporadic.