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On Huppert's Rho-Sigma conjecture

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Joint work with *Z. Akhlaghi* and *S. Dolfi*

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General notation

In this talk, the word “group” will always mean “finite group”.

Given a group G, we denote by $\mathrm{Irr}(G)$ the set of the *irreducible complex characters* of G, and we define the *degree set* of G as

$$\mathrm{cd}(G) = \{\chi(1) \,:\, \chi \in \mathrm{Irr}(G)\}.$$

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Huppert's Rho-Sigma conjecture

In the 80's, Bertram Huppert proposed a remarkable problem concerning the "arithmetical structure" of the degree set:

Is it true that, for a given group G, at least one of the numbers in $\mathrm{cd}(G)$ *is divisible by a "large" portion of the entire set of primes that appear as divisors of some element of* $\mathrm{cd}(G)$*?*

Huppert's Rho-Sigma conjecture

More precisely, for $\chi \in \mathrm{Irr}(G)$, denote by $\pi(\chi)$ the set of prime divisors of the degree $\chi(1)$, and define

$$\rho(G) = \bigcup_{\chi \in \mathrm{Irr}(G)} \pi(\chi),$$

$$\sigma(G) = \max\{|\pi(\chi)| \,:\, \chi \in \mathrm{Irr}(G)\}.$$

- Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that $|\rho(G)| \le f(\sigma(G))$ holds for every group G?
- Does $|\rho(G)| \le 2\sigma(G)$ hold for every *solvable* group?

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The solvable case: a remark

As regards the solvable case, we first note that the bound $|\rho(G)| \leq 2\sigma(G)$ would be optimal *for every value of* $\sigma(G)$.

In fact, for every positive integer n, there exist a set $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ of pairwise distinct primes, and a family of solvable groups $\{G_1, \ldots, G_n\}$, such that $\mathrm{cd}(G_i) = \{1, p_i, q_i\}$ for all $i \in \{1, \ldots, n\}$.

Hence, setting $G = G_1 \times \cdots \times G_n$, we get $|\rho(G)| = 2\sigma(G)$ (with $\sigma(G) = n$).

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The solvable case: some history and a recent result

Some results for the solvable case.

- $|\rho(G)| \leq \sigma(G) \cdot 2^{\sigma(G)}$ (Isaacs; 1986).
- $|\rho(G)| \leq \sigma(G)^2 + 10\sigma(G)$ (Gluck; 1987).
- $|\rho(G)| \leq 3\sigma(G) + 32$ (Gluck, Manz; 1987).
- $|\rho(G)| \leq 3\sigma(G) + 2$ (Manz, Wolf; 1993).
- $|\rho(G)| \leq 3\sigma(G)$ (Akhlaghi, Dolfi, P.; 2021).

The conjectured bound $|\rho(G)| \leq 2\sigma(G)$ has also been verified in some special cases: for instance, when $\sigma(G) \leq 2$, or when all numbers in $\mathrm{cd}(G)$ are squarefree (Gluck; 1985).

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The question originally posed by Huppert for the general case has an affirmative answer: there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $|\rho(G)| \leq f(\sigma(G))$ holds for every group.

The best bound known to date is

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For some special classes of (non-solvable) groups the bound can be improved.

- If G is a non-abelian simple group, then $|\rho(G)| \leq 3\sigma(G)$ (Alvis, Barry; 1991).

But, in their '93 paper, Manz and Wolf ask if $|\rho(G)| \leq 2\sigma(G) + 1$ is the "right" bound for every group G ("strengthened" Rho-Sigma conjecture). Note that, for every $n \geq 1$ it is possible to construct a (non-solvable) group G with $|\rho(G)| = 2n + 1$ and $\sigma(G) = n$.

- Actually, if G is an almost-simple group, or if $\sigma(G) \leq 2$, then $|\rho(G)| \leq 2\sigma(G) + 1$ holds (Tong-Viet; 2015).

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An example (Akhlaghi, Dolfi, P.; 2021)

Recall that, given a prime $p > 5$ and an integer $f \geq 1$, the degree set of the group $\mathrm{PSL}_2(p^f)$ is

$$\left\{1,\ p^f - 1,\ p^f,\ p^f + 1,\ \frac{1}{2}(p^f + \epsilon)\right\}, \quad \text{where } \epsilon = (-1)^{\frac{p^f-1}{2}}.$$

Let $\Pi = \{p_1^{f_1}, ..., p_n^{f_n}\}$ be a set of prime powers with p_i and f_i as above, such that $|\pi(p_i^{f_i} - 1) \setminus \{2,3\}| = |\pi(p_i^{f_i} + 1) \setminus \{2,3\}| = 1$ for every i. We also require that, for distinct r and s in $\{1, ..., n\}$, the intersection of $\rho(\mathrm{PSL}_2(p_r^{f_r}))$ and $\rho(\mathrm{PSL}_2(p_s^{f_s}))$ is $\{2,3\}$.

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Defining $G_\Pi = \mathrm{PSL}_2(p_1^{f_1}) \times \cdots \times \mathrm{PSL}_2(p_n^{f_n})$, we get

$$|\rho(G_\Pi)| = 3n + 2 \quad \text{and} \quad \sigma(G_\Pi) = n + 2,$$

thus $|\rho(G_\Pi)| = 3\sigma(G_\Pi) - 4$.

As a consequence, if $n \geq 4$, the strengthened Rho-Sigma conjecture fails for the group G_Π. (The set $\Pi = \{29, 67, 157, 227\}$ satisfies the above conditions: $|\rho(G_\Pi)|$ is 14, whereas $\sigma(G_\Pi)$ is 6.)

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Let Π_n be a set of size n as above. *Assuming that such a set exists for every* $n \geq 1$ (Hypothesis $(*)$), we get

$$\lim_{n \to \infty} \frac{|\rho(G_{\Pi_n})|}{\sigma(G_{\Pi_n})} = 3.$$

(Primes up to 10^6 yield a group G for which $|\rho(G)|/\sigma(G) > 2.999$).
Furthermore, for every $m \geq 3$, we have a group G (i.e., $G_{\Pi_{m-2}}$) such that $\sigma(G) = m$ and $|\rho(G)| = 3\sigma(G) - 4$. So, the best possible statement that one could hope to prove in this context is the following.

Conjecture

Let G be a group.

- *If* $\sigma(G) \leq 5$, *then* $|\rho(G)| \leq 2\sigma(G) + 1$.
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The following result is a contribution in this direction.

Theorem A (Akhlaghi, Dolfi, P.; 2021)

Let G be a group with trivial Fitting subgroup. Then, denoting by $\pi(G)$ the set of prime divisors of $|G|$, the following conclusions hold.

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Note that, under our hypothesis, we have $\pi(G) = \rho(G)$.

Assuming $(*)$, the bounds in the above statement are best possible for every value of $\sigma(G)$. For $\sigma(G) \in \{1, 2, 3, 4\}$ they are attained by $\mathrm{PSL}_2(2^2)$, $\mathrm{PSL}_2(2^6)$, $\mathrm{PSL}_2(2^{14})$ and $\mathrm{PSL}_2(2^{18})$.

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The following result is a contribution in this direction.

Theorem A (Akhlaghi, Dolfi, P.; 2021)

Let G be a group with trivial Fitting subgroup. Then, denoting by $\pi(G)$ the set of prime divisors of $|G|$, the following conclusions hold.

- *If $\sigma(G) \leq 5$, then $|\pi(G)| \leq 2\sigma(G) + 1$.*
- *If $\sigma(G) \geq 6$, then $|\pi(G)| \leq 3\sigma(G) - 4$.*

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On the proof of Theorem A

The following special case turns out to be a key step.

Theorem (Akhlaghi, Dolfi, P.; 2021)

Let G be a group with trivial Fitting subgroup and with no simple characteristic subgroups. Then $|\pi(G)| \leq 2\sigma(G)$.

- As $\mathbf{F}(G) = 1$, we have $\mathbf{F}^*(G) = S_1 \times \cdots \times S_k$ where the S_i are non-abelian simple groups.
- By induction, we can assume that $\mathbf{F}^*(G)$ is a minimal characteristic subgroup. In particular, the S_i are pairwise isomorphic; setting $K = \bigcap \mathbf{N}_G(S_i)$, also the groups $K/\mathbf{C}_K(S_i)$ are pairwise isomorphic (almost-simple with socle S_i), and K embeds in the direct product of the $K/\mathbf{C}_K(S_i)$.

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Two useful lemmas.

Lemma 1

Let G be a permutation group on the finite set Ω. Then there exist two disjoint subsets Γ, Δ of Ω such that $|G : G_\Gamma \cap G_\Delta|$ is divisible by all the "large" primes, and at least half of the "small" primes, in $\pi(G)$.

Lemma 2

Let G be an almost-simple group with socle S. Then there exist μ, ν in $\mathrm{Irr}(S)$ whose degrees "cover" together $\pi(S)$ except at most one prime, and whose inertia indices in G both cover $\pi(G) \setminus \pi(S)$.

Back to the solvable case

In order to prove that, for every solvable group G, we have $|\rho(G)| \leq 3\sigma(G)$, the following result is critical.

Theorem (Akhlaghi, Dolfi, P.; 2021)

Let G be a solvable group whose Frattini subgroup is trivial. Then there exist χ_1, χ_2 in $\mathrm{Irr}(G)$ and a prime $p \in \{2,3\}$ such that $\rho(G) \subseteq \pi(\chi_1) \cup \pi(\chi_2) \cup \{p\}$.

Also in this situation, a lemma on permutation groups comes into play.

Lemma 3 (Akhlaghi, Dolfi, P.; 2021)

Let G be a solvable permutation group on the finite set Ω. Then there exists a subset Δ of Ω such that $|G : G_\Delta|$ is divisible by all the primes in $\pi(G)$ except possibly one prime $p \in \{2,3\}$.

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Let $\mathbf{E}(G)$ be the subgroup generated by all the components (i.e., non-trivial, quasi-simple, subnormal subgroups) of G. The following result deals with a class of groups including both solvable groups and groups with trivial fitting subgroup.

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